Automata and Computability	**Prelim 1**
CS481 Fall 2005	**September 23**

This is a 50-minute in class closed book exam. All questions are straightforward and you should have no trouble doing them. Please show all work and write legibly. Thank you.

1. For each of the following languages, state whether it is regular or non-regular. In each case, give a convincing justification in one or two sentences. You can use the fact that $\{0^n1^n | n \geq 0\}$ is not regular.

   (a) $L = \{0^{i^2} \mid i \geq 0\}^*$.

   (b) $L = \{0^i w 1^i \mid w \in \{0,1\}^*, i \geq 0\}$.

   (c) $L = \{w\$x \mid w, x \in \{0,1\}^*, \#0(w) = \#1(x)\}$ where $\#0(w)$ is the number of zeros in w.

   (d) $L = \{wx \mid w, x \in \{0,1\}^*, \#0(w) = \#1(x)\}$.

   (e) The language that contains as strings all words from this prelim.

   Answer.

   (a) L contains the strings $0^0 = \varepsilon$ and $0^1 = 0$. Thus, L is actually 0^* and is regular.

   (b) Regular. Again $\{0,1\}^*$.

   (c) $L \cap 0^*\$1^* = \{0^i\$1^i \mid i \geq 0\}$. Using, homomorphism $h(0) = 0$, $h(1) = 1$ and $h(\$) = \varepsilon$, we have $h(L) = \{0^n1^n \mid n \geq 0\}$.

   (d) Regular. This is a bit tricky. But can be seen by just doing a bit of algebra to reduce the given condition to say $|w| = \#1(wx)$ and thus every string has such a prefix.

   (e) Regular as finite.

2. Prove that the set $L = \{ww^R \mid w \in \{0,1\}^*\}$ is not regular using the closure properties of regular languages, and the fact that $\{0^n1^n | n \geq 0\}$ is not regular.

   Answer. Let the homomorphism h be $h(0) = h(\overline{0}) = 0$ and $h(1) = 1$. So $h^{-1}(L) \cap 0^*1^*\overline{0}^* = \{0^i1^{2j}\overline{0}^i \mid i,j \geq 0\} = L_3$. Use the homomorphism $g(0) = 0$, $g(1) = \varepsilon$ and $g(\overline{0}) = 1$ so that $g(L_3) = \{0^n1^n | n \geq 0\}$. Since all these operations preserve regularity, and the final language is not regular, it follows that the given language is not regular.

3. Suppose $L, R \subseteq \Sigma^*$ are regular languages. Define the following operation on two strings: given two strings x and y, $Shuffle(x,y)$ is the set of all strings w such that i) $|w| = |x| + |y|$, and ii) symbols from x and y are interspersed to create w. For instance, if

$x = abb$ and $y = baccb$, then $w = abaccbbb$ is in $Shuffle(x, y)$ but $abbbbacc$ is not, since the symbols of y are not in order. In other words, both x and y are present as *non-overlapping* substrings of w. Using machine construction, show that the following language is regular :

$$\{w \mid \exists x \in L, y \in R, w \in Shuffle(x, y)\}$$

Answer. Let $(Q_1, q_0, \Sigma, \delta_1, F_1)$ and $(Q_2, q_0', \Sigma, \delta_2, F_2)$ be the two DFAs for the languages L and R. Define the new non-deterministic finite automaton which has the set of states $Q_1 \times Q_2$. The idea is that at state (q_1, q_2) on reading a symbol from the input string w, we non-deterministically decide to make the move for either q_1 or q_2. So $\delta((q_1, q_2), a) = \{(\delta_1(q_1, a), q_2), (q_1, \delta_2(q_2, a))\}$. The start state is (q_0, q_0') and the set of final states is $F = F_1 \times F_2$.

We need to prove that the NFA accepts the given set of languages. As usual, we do it by induction. The inductive claim is that on a string w,

$$\widehat{\delta}((q_0, q_0'), w) = \{(q_1, q_2) \mid \exists x, y, w = \text{Shuffle}(x, y) \wedge q_1 = \widehat{\delta_1}(q_0, x) \wedge q_2 = \widehat{\delta_2}(q_0', y)\}.$$

The induction is easy to do. Here is the outline. Base case is trivial since on ε, both coordinates stay at the respective start states. Suppose that we have proved the claim for all strings of length at most n. For a string $w = za$, $|w| = |z| + 1 = n + 1$, we have that

$$\widehat{\delta}((q_0, q_0'), w) = \{(q_1, q_2) = \delta((q, q'), a), s.t. (q, q') = \widehat{\delta}((q_0, q_0'), z)\}$$

Now, the set of all x and y such that $w \in \text{Shuffle}(x, y)$ is given by

$$\{(x', y') \mid w = za \in Shuffle(x', y')\} = \{(xa, y) \mid z \in Shuffle(x, y)\} \cup \{(x, ya) \mid z \in Shuffle(x, y)\}$$

Applying the above definition of $\widehat{\delta}$ and the inductive hypothesis on z, we get the result.

Once we have this inductive claim, by definition of the final states, we can prove our required result.