CS 381	**Introduction to Theory of Computing**	FINAL EXAM
SUMMER 1999		JULY 2, 1999

This exam is open book, open notes, but you may not share books or notes with other students during the exam. During the exam you are not allowed to give or receive help, or to cooperate with anybody else. Write your answers in the exam booklet. Show all your work and clearly indicate your answers. Read the text carefully and make sure you do not overlook details. Write legibly. We reserve the right to ignore illegible answers.

Good luck!

Write your name in the space below and on your booklet after you have read the text above. At the end of the exam, turn in this sheet together with your booklet.

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1. **(20 points)** Let an alphabet, Σ, be given. For any string $x = c_1c_2\cdots c_n \in \Sigma^*$, define the operation $star(x)$ by

$$star(c_1c_2\cdots c_n) = L(c_1^*c_2^*\cdots c_n^*).$$

That is, $star : \Sigma^* \to 2^{\Sigma^*}$ takes a string as input and produces a set of strings, each of which is formed by taking each symbol in the string and replacing it with 0 or more repetitions of that same symbol. Note that the order of the symbols is preserved. So for example, $star(abcd) = L(a^*b^*c^*d^*)$ and $star(\epsilon) = \{\epsilon\}$. We can extend this definition to sets of strings in the following way. For any set of strings $A \subseteq \Sigma^*$, define

$$Star(A) = \bigcup_{x\in A} star(x).$$

So $Star(A)$ is the set of all strings which are members of $star(x)$ for some $x \in A$. So for example, $Star(\{a, b, aba, cc\}) = L(a^*) \cup L(b^*) \cup L(a^*b^*a^*) \cup L(c^*c^*)$. And it turns out that $Star(L((abcd)^*)) = L((a + b + c + d)^*)$.

Prove that if A is regular, then $Star(A)$ is also regular. [Note that the above formula does not directly imply that $Star(A)$ will be regular. Regular sets are *not* closed under *infinite* union.]

2. **(6+10=16 points)** For each of the languages below, decide if it is regular, non-regular but a CFL, or neither regular nor a CFL. Prove your claims.

   (a) $\{a^{2^n} | n \text{ is not prime and } n \bmod 1999 \leq 381\}^*$

   (b) $\{a^m b^n c^p | m \cdot n = p\}$

(turn over)

3. **(12 points)** Consider the grammar G given below:

$$S \rightarrow YX|YS|ZY$$
$$X \rightarrow ZZ|a$$
$$Y \rightarrow ZY|a$$
$$Z \rightarrow XX|b$$

Use the CKY algorithm to determine if *babba* is in $L(G)$. Show your work!

4. **(15 points)** Let an alphabet, Σ, with more than one symbol be given. We have seen in class how to encode Turing Machines as strings over Σ. In a similar way, we could also encode CFG's as strings over Σ. Our encoding would essentially just list all the terminal symbols, the non-terminal symbols, and all the productions in the grammar. The details of this encoding are not important; you may assume any particular syntax you wish, if it helps you.

   Consider the set $X = \{G\#z | z \in L(G), z \neq \epsilon\}$. Here we assume that the G appearing on the left is a suitable encoding of a context-free grammar. Also, assume that $\#$ is a new symbol that is not used in the encoding of any grammar and z is an arbitrary string. X is therefore the set of all grammar-string pairs such that the grammar accepts the string and the string is non-empty. Decide if the set X is recursive, r.e. but not recursive, or not r.e. Argue your claim by describing *at a high level* how a Turing Machine that accepts X would behave, or by providing a reduction or contradiction proof.

5. **(3+5+3+5=16 points)** Answer each of the following questions with "Yes" or "No". Give a brief explanation for parts (b) and (d).

   (a) Given a recursive set, A, and an arbitrary Turing machine, M, is it possible that M loops on every string in A?

   (b) Is it possible for there to be two sets, $A \neq \emptyset$ and $B \neq \emptyset$, such that neither is a subset of the other and yet $A^* = B^*$? Explain why not or give a positive example.

   (c) Let M be a Turing Machine. If $L(M) = \{a^{2^n} | n \geq 0\}$, then does this imply that M is total?

   (d) If C is a CFL, A is a regular language, and $A \cap B = C$, then is it possible that B is a non-CFL? Explain why not or give a positive example.

End of Exam