4.

i) Find a context-free grammar such that $L(G)=\{w \in \{0,1\}^* \mid \#_0(w)=\#_1(w)\}$.

Solution:

$S \rightarrow 0S1 \mid 1S0 \mid SS \mid \varepsilon$

In order to prove this claim, need to show:

a) $\forall \alpha \in (N \cup \Sigma)^*$, if $S \rightarrow_G^* \alpha$ then α satisfies i) $\#_0(\alpha)=\#_1(\alpha)$

b) $\forall \chi$ s.t. $\mid\#_0(\chi)=\#_1(\chi)$ then $S \rightarrow_G^* \chi$

a) Induction on the length of the derivation $S \rightarrow_G^* \alpha$

Basis

If $S \rightarrow_G^0 \alpha$, then α = S. The sentential form trivially satisfies condition i)

Induction Hypothesis

Let $S \rightarrow_G^n \beta$ satisfy condition i)

Induction Step

Suppose $S \rightarrow_G^{n+1} \alpha$. Let β be the sentential form immediately preceding α such that $S \rightarrow_G^n \beta$ $S \rightarrow_G^1 \alpha$. By *I.H.* β satisfies i).

Consider four cases corresponding to 4 possible productions of grammar to derive α from β:

- $S \rightarrow SS \mid \varepsilon$ cases trivial, neither changes number of 1's and 0's $\therefore$ condition i) would still hold for α
- $S \rightarrow 0S1$

  $\exists \beta_1, \beta_2$, $\beta_1, \beta_2 \in (N \cup \Sigma)^*$ s.t. $\beta=\beta_1 S \beta_2$ and $\alpha=\beta_1 0S1 \beta_2$

  $\#_0(\alpha) = \#_0(\beta) + 1$

  $= \#_1(\beta) + 1$ (since β satisfies i)

  $= \#_1(\alpha)$

  so i) holds for α

- $S \rightarrow 1S0$, same as above

Thus, in all cases α meets condition i). This concludes the proof that if $S \rightarrow_G^* \alpha$ then $\#_0(\alpha)=\#_1(\alpha)$.

$S \rightarrow 0S1 \mid 1S0 \mid SS \mid \varepsilon$

b) Induction on the length of $|\chi|$

Basis
If $|\chi| = 0$, we have $\chi = \varepsilon$, and $S \to_G^* \chi$ using $S \to \varepsilon$

Induction Hypothesis
$\forall\ |\chi| \leq n-1$ s.t. $\#_0(\chi) = \#_1(\chi)$, let $S \to_G^* \chi$

Induction Step
Let $|\chi| = n$ then consider two cases:

1. there exists a proper prefix y of χ (one such that $0 < |y| < |\chi|$) satisfying i)
2. no such prefix exists

1) $\chi = yz$ for some z, $0 < |z| < |\chi|$, and z also satisfies i) :

$$\#_0(z) = \#_0(\chi) - \#_0(y) = \#_1(\chi) - \#_1(y) = \#_1(z)$$

By *I.H.* $S \to_G^* y$ and $S \to_G^* z$. We can then derive χ by:

$$S \to_G^1 SS \to_G^* yS \to_G^* yz = \chi$$

2) no such y exists $\therefore$

- $\chi = 1z0$ for some z, and z satisfies i) since:

  $$\#_0(z) = \#_0(\chi) - 1 = \#_1(\chi) - 1 = \#_1(z)$$

  by IH $S \to_G^* z$. We can then derive χ by:

  $$S \to_G^1 1S0 \to_G^* 1z0 = \chi$$

- OR $\chi = 0z1$ for some z… (repeat above)

Thus, every string satisfying i) can be derived. This concludes the proof that if $|\#_0(\chi) = \#_1(\chi)$ then $S \to_G^* \chi$.

Comment: 2 inductions in one problem… what could be more fun? For those of you that used complicated grammars with more productions, keep in mind fewer productions is easier come time for you to prove your claim. For those of you who gave incorrect grammars and then “proved” both directions… obviously your proof cannot be correct.