Problem 4 This problem did not ask for a proof, so we did not take off points for incorrect proofs, informal proofs, or total lack of a proof.

Many of you argued informally to show your example satisfied (b), with the idea that, "the intersection of a finite number n of sets in W looks like such and such, so as n goes to infinity it must be empty." Please see the proofs below to see how to prove this more formally.

Solution 1: Let $W = \{L_i | \ i \in \mathbb{N}\}$, where $L_i = \{w \in \{0,1\}^* | \ |w| > i\}$. Note: with W of this form, we can naturally associate subsets of W with subsets of $\mathbb{N}$: $V \subseteq W$ corresponds to $S_V = \{i | \ L_i \in V\}$. And conversely $S \subseteq \mathbb{N}$ corresponds to $\{L_i | i \in S\}$.

Proof that W satisfies (a): Let V be a finite subset of W, and let S_V be the corresponding finite subset of $\mathbb{N}$. Then S_V contains a maximal element, say n. Then, $1^{n+1} \in L_i$ for all $i \in S_V$. Therefore, $1^{n+1} \in \bigcap_{i \in S_V} L_i = \bigcap_{L \in V} L$, so this intersection is non-empty.

Proof that W satisfies (b) and (c): Let T be any subset of $\mathbb{N}$. Assume there exists some string $x \in \bigcap_{i \in T} L_i$. By definition, this means that $x \in L_i$ for all $i \in T$. Suppose $|x| = k$. Then for $i > k$, $i \notin T$. Thus, T must be finite. It follows that any subset of W with nonempty intersection is finite, so all infinite subsets of W have empty intersection. We have now shown (c). For (b), it suffices to note that there exists an infinite subset of W (in particular, W itself is infinite), and by (c), this subset has empty intersection.

Solution 2: Let $W = \{L_i | \ i \in \mathbb{N}\}$, where $L_i = ((01)^i)^* - \{\varepsilon\}$.

Proof that W satisfies (a): Let S be a finite subset of $\mathbb{N}$, and let $k = \text{LCM}\{i \mid i \in S\}$. Then $\bigcap_{i \in S} L_i = L_k \neq \emptyset$. Note: It is **not** the case that $\bigcap_{i \in S} L_i = L_k$, where k is the greatest number in S (this was a common error).

Proof that W satisfies (b) and (c): Let T be a subset of $\mathbb{N}$. Assume there exists some string $x \in \bigcap_{i \in T} L_i$. By definition, this means that $x \in L_i$ for all $i \in T$. Suppose $|x| = k$. Then for $i > k$, $x \notin L_i$, so T must be finite.

Solution 3:

This solution satisfies only (a) and (b), and is somewhat complicated. However, we point out this solution because several students attempted a solution along these lines.

The idea is to let L_i be the set of all strings in $\{0,1\}^*$ except x_i, where x_i is some string. There were several students who had this idea but didn't explicitly say what x_i should be. Choosing x_i appropriately is essential for satisfying (b); for each string $x \in \{0,1\}^*$ there must be some i such that $x_i = x$. In other words, the function f from $\mathbb{N}$ to $\{0,1\}^*$ such that $f(i) = x_i$ must be onto.

We first define an order on $\{0,1\}^*$ as follows: define $x \lhd y \Leftrightarrow |x| < |y|$ or $(|x| = |y|$, and the first place from the left at which x and y differ has a 0 in x and a 1 in y). Observe that this is a well-ordering; in particular, for every string x, there is a string y such that $x \lhd y$ and there is no z such that $x \lhd z \lhd y$. When this relationship holds, call y the successor of x.

Next, we define a map f from $\mathbb{N}$ to $\{0,1\}^*$ inductively. First, let $f(0) = \varepsilon$. Now assume f(n) is defined for all $n < m$. Then let f(m) be the successor of f(m-1) in $\{0,1\}^*$, with successor as defined above.

Note that f is onto.

Let $x_i = f(i)$. Now, let $W = \{L_i | \ i \in \mathbb{N}\}$, where $L_i = \{0,1\}^* - \{x_i\}$.

We omit the proof that this solution is valid, but (b) holds because f is onto, so that the intersection of all the languages in W is empty.

Some solutions submitted along these lines did not define f explicitly and basically said, "let f be any one-to-one map from $\mathbb{N}$ to $\{0,1\}^*$." The problem with this is that not all such maps are onto. For example, if $f(n) = 0^n$, then the intersection of all languages in W contains any string of all 1's. Similarly, for any such f that is not onto, the intersection of all languages in W is non-empty (it contains any string x not in the range of f), and (b) is not satisfied.