

Which of the following statements holds for every three languages  $L_1, L_2, L_3$ .

i)  $(L_1 \cap L_2)L_3 = L_1L_3 \cap L_2L_3$

FALSE.

Proof by counterexample:

$$L_1 = \{a\}$$

$$L_2 = \{b\}$$

$$L_3 = \{c\}$$

$$(L_1 \cap L_2)L_3 = \{\}\{c\} = \{c\}$$

$$(L_1L_3 \cap L_2L_3) = \{ac\} \cap \{bc\} = \{\}$$

$$\{c\} \neq \{\}$$

ii)  $(L_1^* = L_1^*L_1^*$

TRUE.

To prove this claim, we must prove a:  $(L_1^* \subseteq L_1^*L_1^*$  and b:  $(L_1^*L_1^* \subseteq L_1^*$ .

a) Take any  $x \in L_1^*$  and show  $x \in L_1^*L_1^*$ . Observe  $x = x\epsilon$ .  $x \in L_1^*$  by assumption and  $\epsilon \in L_1^*$  by definition of  $L_1^*$  therefore  $x \in L_1^*L_1^*$ .

b) Take any  $x \in L_1^*L_1^*$  and show  $x \in L_1^*$ . Observe  $x \in L_1^*L_1^*$  means  $x = yz$  where  $y \in L_1^*$  and  $z \in L_1^*$ . By definition of  $L_1^*$  this means that  $y \in L_1^{\{n\}}$  and  $z \in L_1^{\{m\}}$  for some non-negative integers  $n$  and  $m$ . This means that  $yz \in L_1^{\{n+m\}}$ , which in turn means that  $x = yz \in L_1^*$  because by definition  $L_1^{\{n+m\}} \in L_1^*$ .

iii)  $(L_1 \cup L_2) \cap L_3 = L_1 \cup (L_2 \cap L_3)$  FALSE.

Proof by counterexample:

$$L_1 = \{a\}$$

$$L_2 = \{b\}$$

$$L_3 = \{c\}$$

$$(L_1 \cup L_2) \cap L_3 = \{a, b\} \cap \{c\} = \{\}$$

$$L_1 \cup (L_2 \cap L_3) = \{a\} \cup \{\} = \{a\}$$

$$\{\} \neq \{a\}$$