CS381 Fall 2000	Handout 37
December 1, 2000	

Practice problems for the Final I.

Problem 37.1. Design a Turing Machine to recognize palindromes.

Problem 37.2. Design a Turing Machine to compute $f(n) = 2n$.

Problem 37.3. Which of the following problems is decidable? Why?

a) Given a TM M and a string y, does M ever write the symbol $\sharp$ on its tape on input y?

b) Given a context free grammar G over $\{a, b\}$, does G generate all the strings of the language $\{a, b\}^*$ of length ≤ 381?

c) Given a context free grammar G over $\{a, b\}$, does G generate all the strings of the language $\{a, b\}^*$ of length greater than 381?

d) Given a TM M, are there infinitely many TMs M' accepting the same r.e. set $A = L(M)$?

e) Given a TM M and a string y, does M accept y?

Solution 37.3.

a) Undecidable, since the halting problem is reducible to it. Indeed, given M (without $\sharp$ among the tape alphabet) build M' as follows: M' simulate M but each time M wants to halt M' first prints $\sharp$ and then halts. It is clear, that M halts on y if and only if M' writes $\sharp$ on input y. Therefore, if we could decide (a) we would be able to decide the halting problem.

b) Decidable. Each individual problem $x \in L(G)$ is decidable, and we have to check whether each string of length ≤ 381 (a given finite set of them) is in $L(G)$.

c) Undecidable. Otherwise we could decide $L(G) = \{a, b\}^*$ by deciding (b) and (c) together. Strictly speaking, we are using here a property: an intersection of decidable sets is decidable.

d) Decidable. This is a trivial property, since for each TM M there are infinitely many TMs M' accepting the same r.e. set $A = L(M)$.

e) Undecidable. Otherwise we could be able to decide the problem 'does M accepts ϵ?'. The latter is undecidable by the Rice Theorem, since it corresponds to a nontrivial property of r.e. sets $\epsilon \in L(M)$.

Problem 37.4. Which of the following sets is r.e.? Why?

a) $\{M \mid M$ takes more than 381 steps on some input$\}$

b) $\{M \mid M$ takes ≤ 381 steps on some input$\}$

c) $\{M \mid M$ takes fewer than 381 steps on all inputs$\}$

d) $\{M\sharp x \mid M$ accepts x$\}$

e) $\{M\sharp x \mid M$ does not accept x$\}$

f) $\{\varphi \mid \varphi$ is a false sentence of arithmetic$\}$

Solution 37.4

a) Decidable (cf. Kozen p.236), therefore, r.e.

b) Decidable, therefore, r.e. The argument is similar to (a). It suffices to try 381 steps of M on each of the given finite set of all inputs of length ≤ 381. If M halts at least once, then the test result is "yes". Otherwise, the result is "no", since for all other inputs it takes longer then 381 steps just to read them.

c) Decidable, as the complement to (a), therefore, r.e.

d) Semidecidable (r.e.), since applying M to x provides a positive test for this set.

e) Not r.e. Indeed, its complement is (d), which is r.e., but not decidable, by 37.3e.

f) Not r.e. We can reduce $Th(N)$ to (f) by $R(\varphi) = \neg\varphi$. Indeed, φ is true $\Leftrightarrow$ $\neg\varphi$ is false in arithmetic.

Problem 37.5. Prove that $f(n) = 1 + 2 + \ldots + n$ is primitive recursive. Hint: assume that functions $x + y$, $x \cdot y$, x^y, $x \dot{-} y$, $\mathbf{sign}(x)$, $\mathbf{compare}_{\leq}(x, y)$ are p.r.

Solution 37.5 A primitive recursive scheme for $f(x)$ is

$$\begin{cases} f(0) = 0 \\ f(x+1) = f(x) + x + 1 \end{cases}$$

Here the function $g = 0$, $h(x, y) = y + \mathbf{s}(x)$.

Problem 37.6. Prove that $rm(x) =$"remainder of x devided by 2" is primitive recursive.

Solution 37.6. A primitive recursive scheme for $rm(x)$ is

$$\begin{cases} rm(0) = 0 \\ rm(x+1) = 1 \dot{-} rm(x) \end{cases}$$

Here the function $g = 0$, $h(x, y) = 1 \dot{-} y$.

Problem 37.7. Prove that

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ \textit{undefined} & \text{if } n \text{ is odd} \end{cases}$$

is μ-recursive.

Solution 37.7. $f(x) = 1 + \mu y.(y + rm(x) = 0)$. If x is even, then $rm(x) = 0$, $g(x) = \mu y.(y + rm(x) = 0)$ is equal to 0, and $f(x) = 1$. If x is odd, then $rm(x) = 1$, $g(x) = \mu y.(y + rm(x) = 0)$ is undefined, so is $f(x)$.

Problem 37.8. Find the normal form of the λ-term $(\lambda fgx.f(gx))(\lambda x.x)(\lambda x.x)$.

Solution 37.8. $(\lambda fgx.f(gx))(\lambda x.x)(\lambda x.x) \xrightarrow{\alpha} (\lambda fgy.f(gy))(\lambda x.x)(\lambda x.x) \xrightarrow{\beta} (\lambda gy.(\lambda x.x)(gy))(\lambda x.x) \xrightarrow{\beta} (\lambda gy.gy)(\lambda x.x) \xrightarrow{\beta} (\lambda y.(\lambda x.x)y) \xrightarrow{\beta} \lambda y.y$

Problem 37.9. Design a Post system to recognize palindroms.