CS 322 Final Exam

Friday 18 May 2007—150 minutes

Problem 1: Toolbox (25 pts)

For all of the parts of this problem, you are limited to the following sets of tools:

(A) Runge-Kutta 4/5 Method	(H) Newton's Method
(B) Condition Number	(I) QR Factorization
(C) Secant Method	(J) Midpoint Method
(D) Euler Method	(K) Bisection
(E) Linear Interpolation	(L) SVD
(F) LU Factorization	(M) Backward Euler Method
(G) Monte Carlo	(N) χ^2 Distribution

a. (14 pts) For each of the following problem classifications, list all the tools from the above set that can be used to solve that type of problem.

   (i) Computing low-rank approximations to a data matrix

   (ii) Solving initial-value ordinary differential equations

   (iii) Finding least-squares solutions to linear systems

   (iv) Fitting data with parametric models

   (v) Estimating uncertainty in computed solutions

   (vi) Finding roots of nonlinear scalar equations

   (vii) Performing Principal Components Analysis

   (viii) Solving linear systems

b. (11 pts) For each of the following problem descriptions, choose the most appropriate tool to solve the problem. Also list the other tools that are capable of solving the problem and explain (in one or two sentences) why your choice is more appropriate.

   (i) You are working at NASA, monitoring a satellite that has been measuring the solar wind daily for the last year (storing its current position, time, and velocity of the measured solar wind). You need a model that predicts solar wind velocity from these parameters that is quadratic with respect to distance from Earth and linear in time.

(ii) You are still working at NASA, only the power source on the satellite failed in the middle of a crucial orbital maneuver. Your superiors want a prediction on where the satellite will crash into earth (given its known position and velocity, and assuming a simple model of air resistance and weather).

(iii) Before reporting your computed impact location to the news media, you'd better find out how accurately you know the location. The important sources of uncertainty in the problem are the satellite's initial velocity and the parameters of the atmospheric model.

(iv) Surprisingly, you are still gainfully employed at NASA. After recovering the wreckage, you want to figure out what went wrong by analyzing the chemical makeup of the battery. You extract some material, vaporize it, heat the gas, and measure the spectrum of light emitted from it. You know what gases will be present in the mixture, and you know what their spectra look like individually, just not the amounts of each gas. You're careful to arrange the experiment so that the spectra will add together. You want to estimate the amount of each gas component in the mixture given the above information.

(v) You can't get the fit in (iv) to work; the residual is unreasonably high. There must be something in the sample you don't know about. You repeat the spectral analysis process for samples drawn from 25 different locations in the battery, and you want to identify what materials might be varying in concentration without assuming you know what components are present ahead of time.

The answers are not necessarily uniquely determined; the important thing is that you support your choice with a plausible and correct argument.

Problem 2: MATLAB Code (14 pts)

Write efficient vectorized MATLAB code for each of the following expressions. Assume the cost of a matrix multiplication is independent of the contents of the matrices.

a. $y_i = D_{i,i}x_i$ for $i = 1\ldots n$, where $\mathbf{D}$ is diagonal

b. $C_{3,5} = \sum_{k=1}^{n} A_{3,k}B_{k,5}$

c. $B_{i,j} = \sum_{k=1}^{5} \sigma_k U_{i,k}V_{j,k}$ for all $i = 1\ldots m$ and $j = 1\ldots n$, where $\mathbf{U}$ is an $m \times n$ matrix, $\mathbf{V}$ is an $n \times n$ matrix, and $\mathbf{B}$ is an $m \times n$ matrix.

d. $B_{i,j} = \sum_{k=1}^{n} M_{i,k}A_{k,j}$ for all $i = 1\ldots n$ and $j = 1\ldots n$, where $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ and $\mathbf{M}$ is $n \times n$ and has the following structure:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & m_{1,4} & 0 & \ldots & 0 \\ 0 & 1 & 0 & m_{2,4} & 0 & \ldots & 0 \\ 0 & 0 & 1 & m_{3,4} & 0 & \ldots & 0 \\ 0 & 0 & 0 & m_{4,4} & 0 & \ldots & 0 \\ 0 & 0 & 0 & m_{5,4} & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & m_{n,4} & 0 & \ldots & 1 \end{bmatrix}$$

e. $B_{i,j} = \sum_{k=1}^{n} Q_{i,k}A_{k,j}$ for all $i = 1\ldots n$ and $j = 1\ldots n$, where $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, $\mathbf{Q} = \mathbf{I} - \mathbf{u}\mathbf{u}^T$ and the first four entries of $\mathbf{u}$ are all zero.

f. $c_i = \sum_{k=1}^{n} A_{i,k}b_k$, where $\mathbf{b}$ and $\mathbf{c}$ are n-vectors and $\mathbf{A}$ is $n \times n$ and block diagonal with the $k \times k$ matrix $\mathbf{M}$ repeated on its diagonal, i.e. $\mathbf{A}$ is of the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{M} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{M} \end{bmatrix}$$

g. $c_i = \sum_{k=1}^{n} A_{i,k}x_k$ for $i = 1\ldots n$, where $\mathbf{A}$ is $n \times n$ and $\mathbf{x}$ is an n-vector that is known to have only 3 nonzero entries (but it's not known ahead of time which ones are nonzero)

Problem 3: Data Fitting (15 pts)

Consider the following plot of data points:
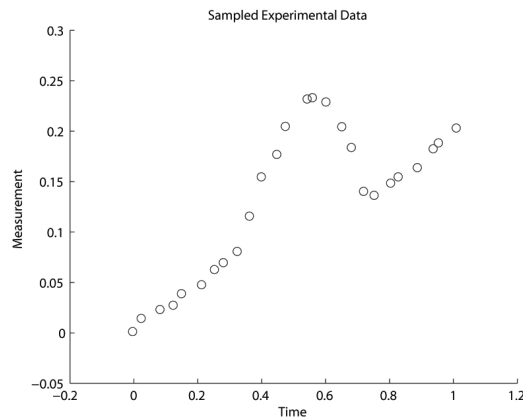

and the following possible models:

1. linear

2. polynomial

3. spline

for fitting the measurement y_i as a function of the time t_i.

a. (4 pts) Which of the models above would best fit this data?

b. (4 pts) Explain (in one to two sentences each) what problems would occur with the other two models

c. (4 pts) Set up the systems that you would need to solve for both a linear and a cubic polynomial fit.

d. (3 pts) How would your systems in part (c) change if each y_i had a different uncertainty σ_i?

Problem 4: Linear transformations and systems (20 pts)

Consider the 2×2 matrices A, B, and C, which transform the unit circle into the following three ellipses (the dot shows where a particular point is mapped to by each of the three transformations):
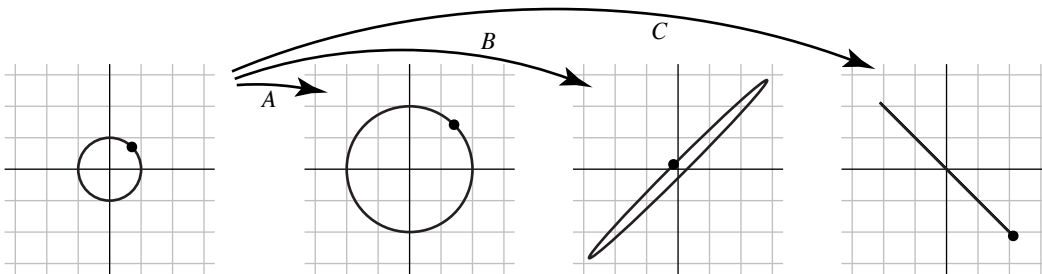
    a. (8 pts) Write down the SVD of each of these matrices, choosing from the following matrices for U, Σ, and V:

$$\text{(i)} \ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{(ii)} \ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{(iii)} \ \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{(iv)} \ \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{(v)} \ \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

Consider the three matrix factorizations LU, QR, and SVD, each of which can be used to solve square or overdetermined linear systems.

    b. (6 pts) Which is the fastest usable factorization to solve a 4×4 square system for each of the following three sets of singular values?

       (i) $\text{diag}(\Sigma) = \begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix}$

       (ii) $\text{diag}(\Sigma) = \begin{bmatrix} 10^4 & 10^3 & 10^2 & 10^{-1} \end{bmatrix}$

       (iii) $\text{diag}(\Sigma) = \begin{bmatrix} 4 & 3 & 2 & 0 \end{bmatrix}$

    c. (6 pts) Which is the fastest usable factorization to solve an 8×4 overdetermined system for each of the three sets of singular values from part (b)?

Assume computations are done with IEEE single-precision arithmetic.

Problem 5: Convergence (10 pts)

To learn about the error in our computed solution to an ordinary differential equation, we used three different methods to integrate the equation through a fixed time interval using a range of different step sizes. For each run we measured the error relative to a known analytic solution. The results are shown (schematically) in the following logarithmic plots:
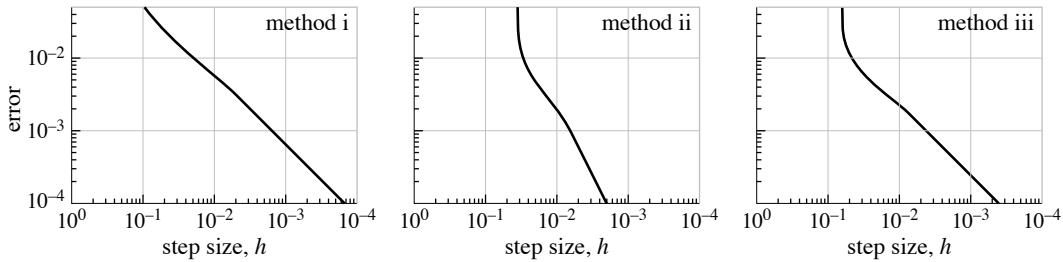
a. (2 pts) Order the three methods from most to least stable.

b. (2 pts) What is the order of accuracy of each method?

c. (2 pts) Which methods are these, from among the ones we studied in class?

We did much the same thing with three methods for solving a nonlinear equation in one variable. In this case the measure of error was the residual function value, and we calculated it for increasing numbers of iterations and plotted it semilogarithmically:
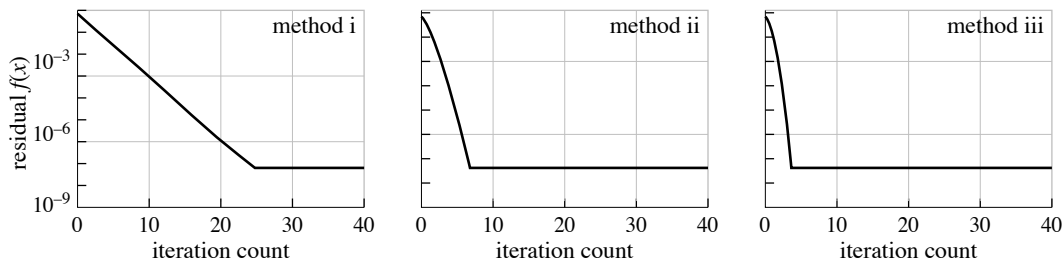
d. (2 pts) Classify the convergence of each method as linear, sublinear, or superlinear.

e. (2 pts) Which methods are these, from among the ones we studied in class?

Problem 6: Statistics of Gaussians (8 pts)

For all three parts below, your answers can be in terms of the MATLAB functions `chi2pdf`, `chi2cdf`, and `chi2inv`.

    a. (4 pts) Suppose $\mathbf{x} \in I\!R^k$ is distributed according to a unit variance gaussian. Give a 90% confidence region for $\mathbf{x}$.

Suppose we have time series data $y_1, \ldots, y_m$ that we are approximating using a set of k orthonormal vectors (perhaps obtained from principal components analysis of some data collected earlier). That is, the time series $\mathbf{y}$ is approximated by a linear combination of k basis functions:

$$\mathbf{y} \approx \sum_{j=1}^{k} a_j \mathbf{b}_j.$$

The parameters are $a_1, \ldots, a_k$. The data all have Gaussian errors with standard deviation 1.

    b. Give a 90% confidence region for each of the individual fitted parameters?

    c. Give a 90% confidence region for the parameters jointly.

Problem 7: Condition numbers (8 pts)

Consider the matrix-vector multiplication:

$$\begin{bmatrix} 0 & 4 & 1 \\ 2 & 2 & 1 \\ 1 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$
$$\mathbf{A} \qquad \mathbf{x} \quad = \quad \mathbf{b}$$

The condition number of $\mathbf{A}$ is 22.2, and its singular values are $\sigma_1 = 7.1$, $\sigma_2 = 1.8$, $\sigma_3 = 0.32$.

    a. If I change $\mathbf{x}$ by adding a vector of norm 0.1 and then do the matrix multiplication to recompute $\mathbf{b}$, what is the maximum relative norm change in $\mathbf{b}$ that could result?

    b. If I change $\mathbf{b}$ by adding a vector of norm 0.1 and then solve the linear system to recompute $\mathbf{x}$, what is the maximum relative norm change in $\mathbf{x}$ that could result?

    c. What bounds would the condition number give for both questions above?