CS 322 Prelim 1

Tuesday 20 February 2006—90 minutes

Problem 1: Linear Systems (30 pts)

Here is a linear system at some intermediate step of one of the linear system solution methods we've looked at.

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 10^{-10} & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

Assume that there is nothing special about the original matrix (no special pattern of zeros, for example).

1. What method are we looking at? How did you tell?
2. What iteration is the algorithm about to start?
3. Which element should be used as the pivot in this iteration? Why?
4. Compute the matrix after the iteration is complete.
5. What data other than this matrix would the algorithm need to modify during the iteration, and how?

Problem 2: Problem classification (28 pts)

Consider the following numerical problems:

1. Solving $\mathbf{MQ} = \mathbf{P}$ for $\mathbf{M}$, where $\mathbf{Q}$ and $\mathbf{P}$ are both 4×4.
2. Finding an equation of the form $y = ax^3 + bx^2 + cx + d$ to fit 4 points (x_i, y_i).
3. Finding the value of some function $f(x)$ at x', given values $y_1 = f(x_1), y_2 = f(x_2), ...$ where $x_1 < x_2 ... < x_n$ and $x_1 \leq x' \leq x_n$.
4. Finding values for n variables x_i that best satisfy n linear equations.
5. Finding an equation of the form $y = ae^{bx} + c$ to fit five points (x_i, y_i).
6. Finding a linear transformation of the plane (a 2×2 matrix) to match 3 pairs of points.
7. Finding a rotation of the plane to match 3 pairs of points.
8. Finding values for n variables x_i that best satisfy m linear equations, $m > n$.

Match each problem to the one of the following methods that is **best suited** to solving the problem, and give a one-sentence explanation for each.

(a) solving a system of linear equations;

(b) solving a linear least squares system;

(c) interpolation;

(d) none apply.

Problem 3: Least squares (21 pts)

Here are two least squares problems in the standard form $\mathbf{Ax} \approx \mathbf{b}$:

$$1: \quad \mathbf{A} = \begin{bmatrix} 0 & -1 \\ 2 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \quad ; \quad \mathbf{b} = \begin{bmatrix} -3 \\ 7 \\ -3 \\ -1 \end{bmatrix}$$

$$2: \quad \mathbf{A} = \begin{bmatrix} 1 & 5 \\ 2 & 4 \\ 3 & 3 \\ 4 & 2 \\ 5 & 1 \end{bmatrix} \quad ; \quad \mathbf{b} = \begin{bmatrix} -3 \\ 3 \\ -3 \\ 9 \\ 9 \end{bmatrix}$$

1. What does it mean for a vector $\mathbf{x}$ to be the solution to a linear least squares system in this form?
2. Is $\mathbf{x} = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$ the solution to system 1? Explain how you can tell.
3. Is $\mathbf{x} = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$ the solution to system 2? Explain how you can tell.

Problem 4: Snowfall (21 pts)

After the recent snowfall in Ithaca, we gathered a bunch of data from friends around the area who measured the depth of the snow in their back yards. That is, for a set of n locations (x_i, y_i) (denoting longitude x and latitude y), we have a measurement of the snowfall s_i received at that point. Now we wish to find a model to predict snowfall based on position.

1. Suppose we think that snowfall all across Ithaca for the most recent storm can be predicted by a single linear function in x and y. Set up the system that you would need to solve to find this linear function. Be specific about what variables you are solving for and the structure of your matrices.

2. Suppose we think a single quadratic function in x and y would be a better fit. Can we still compute it with linear fitting? Why or why not?

3. Suppose our data was structured in such a way that we could easily use some form of linear interpolation between the closest sample points, and that we had a large number of well-spaced sample points.

   (a) Give a suitable constraint on the structure of the input data that will make this operation easy.

   (b) Interpolation might produce predictions that are more or less accurate than the functions you fit in (1) and (2). Give two arguments, based on different assumptions, the first claiming that interpolation is better, and the second claiming that a low-order fit is better. Be specific about your assumptions.