# CS280, Spring 2003: Final

This exam is out of 75; the amount for each question is as marked. Clearly explain your reasoning. (Remember, we can't read your mind! Besides, it gives us a chance to give you partial credit.) Don't forget to put your name and student number on each blue book you use. You don't have to do the questions in order; just make sure you mark the question number clearly.

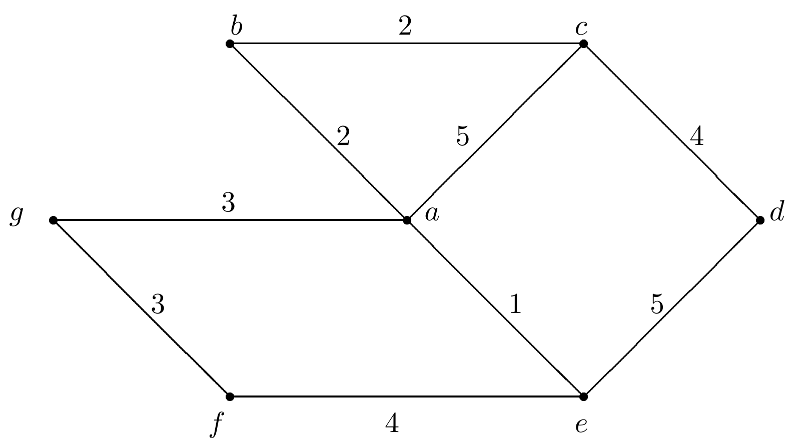
Exams should be graded by tomorrow, and will probably be available for you to pick up starting Monday. You can pick up your exam in 4146 Upson. I'll post a message on netnews and the web when they're available; please don't come before I post the message.

Good luck!

1. [4 points] Prove that $A \cap (A \cup B) = A$.

2. [5 points] Suppose that $a_0 = 2$ and $a_{n+1} = 2a_n + 2$. What are $a_1$, $a_2$ and $a_3$? Guess a formula for $a_n$ (that does not involve $a_{n-1}$) and prove that your guess is correct.

3. [4 points] Draw a minimum spanning tree for the graph below. List the order in which edges are added to the minimum spanning tree.




4. [6 points] Consider the graph above for question 3, ignoring the weights.

   (a) [2 points] Does the graph have an Eulerian path? Explain why or why not. If it does have an Eulerian path, what is it?

   (b) [2 points] Does the graph have an Eulerian cycle? Explain why or why not. If it does have an Eulerian cycle, what is it?

   (c) [2 points] What is a Hamiltonian path? Does the graph have one? If so, describe it.

5. [8 points]

   (a) [3 points] How many functions are there from $\{a, b, c\}$ to $\{1, 2, 3, 4\}$.

   (b) [3 points] How many of these functions are one-to-one?

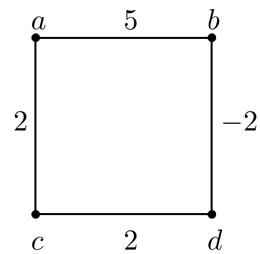
(c) [2 points] How many of these functions are onto?

6. [5 points] Consider the graph below.

   (a) [1 point] Just eyeballing the graph, what is the shortest path from $a$ to $d$ and what is its length?

   (b) [3 points] Compute the shortest path from $a$ to $d$ and its length using Dijkstra's algorithm. Show your work.

   (c) [1 point] Why are the answers in (a) and (b) different?




7. [4 points] How many solutions $(x_1, x_2, x_3, x_4)$ are there to the equation $x_1 + x_2 + x_3 + x_4 = 35$, where $x_1, x_2, x_3, x_4$ are natural numbers? (Just write down the combinatorial expression that describes the answer; there's no need to calculate it numerically.)

8. [4 points] How many integers between 1 and 1000 are divisible by either 8 or 12?

9. [4 points] Show that among any group of five (not necessarily consecutive) integers, there must be two whose difference is divisible by four. [Hint: think of the remainder of each of the numbers when you divide by four.]

10. [3 points] A group of 10 women and 9 men are in a room. If five of the 19 are selected at random, what is the probability that all five are of the same sex? (Again, just write down the combinatorial expression that describes the answer; there's no need to calculate it numerically.)

11. [1 points] When we write $\Pr(X = 3)$, where $X$ is a random variable on a sample space $S$, which event's probability is being calculated?

12. [3 points] What is the probability that a biased coin, with probability 2/3 of landing heads, lands heads 4 times out of 5 flips? (Write this as a fraction in lowest terms.)

13. [5 points] Let $E$ be the event that a randomly generated bit string of length three (that is, a sequence of 0s and 1s of length 3) contains an odd number of 1s, and let $F$ be the event that the bit string starts with a 1. Are $E$ and $F$ independent? If so, prove it. If not, explain why.

14. [5 points] Provide an example that shows that the variance of the sum of two random variables is not necessarily equal to the sum of their variances. More precisely, define a sample space $S$, a probability Pr on $S$, and two random variables $X$ and $Y$ on $S$ such that $Var(X + Y) \neq Var(X) + Var(Y)$. [Hint: make your life (and the graders' lives) easier by taking the sample space to have two elements.]

15. [4 points] Are the formulas $p \Rightarrow (q \Rightarrow r)$ and $(p \Rightarrow q) \Rightarrow r$ equivalent? (If you think they are, then show that their truth tables are identical. If you think they are not, give a truth assignment that shows they are different.)

16. [6 points] Which of the following formulas is true if the domain is the natural numbers, and which are true if the domain is the real numbers. (Explain your answer in each case.)

    (a) $\exists x \exists y (x < y \wedge \forall z (z \leq x \vee y \leq z))$

    (b) $\exists x \exists y (2x - y = 4 \wedge 2x + y = 6)$

    (c) $\exists y \forall x ((x + y)^2 = x^2 + y^2)$

17. [4 points] Suppose that $T(x, y)$ means "student $x$ takes course $y$". Translate each of the following sentences into first-order logic:

    (a) No student takes all courses.

    (b) Bob takes a course only if Ann takes the course.

    (c) Every student takes some course.

    (d) There is a course that no one is taking.