1 Question 1

(a) Find the radius and diameter of K_5

K_5 is the complete graph on 5 vertices. The minimum and maximum distance between any two vertices in a complete graph is 1. Therefore both the radius and diameter of the graph are 1.

(b) Find the radius and diameter of $K_{4,7}$.

$K_{4,7}$ is the complete bipartite graph on 4- and 7-vertex partitions. In a bipartite graph, each vertex v has a radius of 2 (any other vertex in the same partition as v is distance 2 away). Therefore the radius and diameter of the graph is 2.

(c) Find the radius and diameter of Q_4.

Q_4 is the 4-dimensional hypercube. Any two vertices can differ in at most 4 positions (in the traditional sense that the vertices are labelled with binary numbers of length 4), in fact, for a vertex v there exists exactly one vertex w that differs from v in 4 positions, and is hence distance 4 away. So the radius and diameter of the graph is 4.

(d) Find the radius and diameter of C_7.

C_7 is the cycle on 7 vertices. The radius of every vertex in the graph is 3, as you never have to go more than 3 vertices around the cycle in either direction to reach all the other vertices. Therefore, the radius of the graph is 3, and so is the diameter.

2 Question 2

What is the relation between the radius (R) and the diameter (D) of a connected graph? In other words, the goal is to find constants c_1 and c_2 such that $c_1 R \leq D \leq c_2 R$ holds for any connected graph and such that c_1 is as large as possible and c_2 is as small as possible.

First notice that the diameter D cannot be less than the radius R. If $R > D$, then every vertex v would have to have radius $\geq R$, ie the maximum distance between v and another vertex would be $\geq R$ for every vertex v. But then the diameter would be at least R, a contradiction to $R > D$. Therefore $R \leq D$, so $c_1 \geq 1$. Furthermore, we have cases (see question 1) where $R = D$, and so $c_1 \leq 1$, making $c_1 = 1$ necessary.

Now consider a graph with diameter j. This means that there exist at least two vertices, v and w say, such that the minimum distance between v and w is j. Then for any vertex on the shortest path between v and w, its radius is at least $\lfloor \frac{j+1}{2} \rfloor$ (and $\exists$ vertex with radius $\lfloor \frac{j+1}{2} \rfloor$). For any vertex not on the path between v and w, the radius is also greater than $\lfloor \frac{j+1}{2} \rfloor$, due to the path between v and w of length j being the shortest path between the two. Therefore, for a diameter j, the smallest R can be is $\lfloor \frac{j+1}{2} \rfloor$. Thus taking $c_2 = 2$ we guarantee that $D \leq c_2 R$:

$$\begin{aligned} R &\geq \lfloor \frac{j+1}{2} \rfloor \\ 2R &\geq 2\lfloor \frac{j+1}{2} \rfloor \\ &\geq j = D \end{aligned}$$

A graph for which the bound on c_2 is tight is a straight path on n vertices. I.e. two end vertices of degree one, and all intermediate vertices of degree two. Then the diameter of the graph is $n-1$ (the distance between the two end vertices), while the radius of the middle vertex is $\lfloor \frac{n}{2} \rfloor$. If n is odd, say 5, then the diameter is 4, while the radius is 2, and our bound is tight.

3 Question 3

How many nonisomorphic subgraphs does Q_2 have? Recall that a graph is a subgraph of itself and that, by definition, a graph must have at least one vertex.

There are 12 nonisomorphic subgraphs to Q_2:

* There is one nonisomorphic subgraph on one vertex.
* There are two nonisomorphic subgraphs on two vertices (one component graph, two component graph).
* There are three nonisomorphic subgraphs on three vertices (one component graph, two components, three components).
* There are six nonisomorphic subgraphs on four vertices (two with one component, two with two components, one with three components, one with four components).

4 Question 4

Claim: In any finite simple graph with at least two vertices there are always two vertices with the same degree. Either prove the claim true or give a counterexample.

The claim is true. Proof by contradiction.

Assume the claim is not true, i.e. all vertices have different degree. Notice that in a graph on n vertices, the minimum degree is 0 and the maximum degree is $n-1$. Therefore, for n vertices to have n different degrees, they must have degrees $0, 1, \ldots, n-1$. But the vertex with degree $n-1$ is connected to all the other vertices in the graph, which contradicts the assumption that there is a vertex of degree 0. Therefore, there must be at least two vertices of the same degree.

5 Question 5

(a) Can 9 segments be drawn in the plane so that each intersects exactly 3 others? Prove your answer.

Transform the problem into the following graph G:

* For each of the 9 segments create a vertex v_i.
* Put an edge between v_i and v_j if segment i intersects segment j.

So our question is reformulated to asking if it is possible to draw G such that each vertex has degree exactly 3. But with 9 vertices, we note that the total degree of the graph, $9 \times 3 = 27$, is odd, which is not possible in our graph (even if we split it into components). Therefore it cannot be drawn.

(b) Suppose G has 10 vertices, each of degree 5. Show that G is not planar.

Since each vertex has degre 5, any component of the graph would have at least 6 vertices, and so our graph must have one component of all 10 vertices, i.e. G is connected. We have a corrollary on p 505 of the text that states if G is a connected planar simple graph then $e \leq 3v - 6$. But in our case

$$\begin{aligned} e &= \frac{1}{2}\sum_{i=1}^{10} deg(v_i) = \frac{1}{2}\sum_{i=1}^{10} 5 = 25 \\ 3v - 6 &= 3 \times 10 - 6 = 24 \end{aligned}$$

So $e > 3v - 6$, so it is not possible for G to be a planar graph.

6 Question 6

Some friends of yours are working on wireless networks consisting of n mobile devices. Each user reports on who they can communicate with at noon on a particular day. Alice reports that her device can communicate directly with only one other device. Bob reports that his device can communicate directly with 19 other devices. All the remaining users report that each remaining device can communicate directly with exactly 20 other devices. Prove that there exists a communication path between Alice and Bob.

Convert this problem into a graph, where the n people in the network are represented by n vertices $v_1, \ldots, v_n$. Put an edge between two vertices v_i and v_j if person i and person j can communicate directly. And we would like to prove that there exists a path between the vertex representing Alice (v_1 say) and the vertex representing Bob (v_2 say). With this model we have that

$$\begin{aligned} deg(Alice) &= 1 \\ deg(Bob) &= 19 \\ deg(v_i) &= 20 \ \forall i = 3 \ldots n \end{aligned}$$

By way of contradiction, assume that Alice and Bob are not connected, i.e. that they are in different components of the graph. Without loss of generality suppose Alice is in component 1 and Bob is in component 2. Then in component 1 we sum up the degrees:

$$\sum deg(v_i) = 1 + c_1 20$$

which is obviously an odd number (for c_1 some constant). Similarly, the sum of the degrees in Bob's component is also odd ($19 + c_2 20$). In any connected component of a graph it is necessary for the sum of the degrees to be even, and thus Alice and Bob being in different components is not possible, a contradiction.