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Solution Set Homework 8

PART A:

1. Consider the set of strings using letters of the alphabet. For each of the relations, below determine if the relation is reflexive, irreflexive, symmetric, antisymmetric, and or transitive

    a. {(a,b) | a and b have no vowels in common}
    *Not Reflexive, nor Irreflexive – counter example for Reflexive: let* $\alpha$ = *'aeiou'; counter example for Irreflexive: let* $\alpha$ *=' x'.*
    *Symmetric –* $\forall(\alpha, \beta) \in R$ *implies* $(\beta,\alpha) \in R$ *if* $\alpha$ *has different vowels than* $\beta$ *then* $\beta$ *will have different vowels from* $\alpha$
    *Not Antisymmetric -* $\forall(\alpha, \beta) \in R$ *and* $(\beta,\alpha) \in R$ *does not imply that* $\alpha = \beta$*; counter example: let* $\alpha$*= 'ab' and* $\beta$*= 'e'*
    *Not transitive- by counter example: let* $\alpha$ *= dead,* $\beta$ *= cstrifourtiin,* $\chi$ *= beef (the e is in common)*

    b. {(a,b) | a and b have different lengths}
    *Irreflexive – because* $\forall \alpha \in R$ *implies* $(\alpha,\alpha) \notin R$ *(the same word will always have the same number of letters)*
    *Symmetric – for all* $(\alpha, \beta) \in R$ *implies* $(\beta,\alpha) \in R$ *if* $\alpha$ *has a different length than* $\beta$ *then* $\beta$ *will have a different length from* $\alpha$
    *Not Antisymmetric -* $\forall(\alpha, \beta) \in R$ *and* $(\beta,\alpha) \in R$ *does not imply that* $\alpha = \beta$*; counter example: let* $\alpha$*= 'ab' and* $\beta$ *= 'e'*
    *Not transitive- by counter example: let* $\alpha$ *= dead,* $\beta$ *= cstrifourtiin,* $\chi$ *= beef* $(\alpha,\beta) \in R \wedge (\beta,\chi) \in R$ *does not imply* $(\alpha,\chi) \in R$

    c. {(a,b) | a and b end with the same letter}
    *Reflexive – because* $\alpha \in R$ *implies* $(\alpha, \alpha) \in R$ *(the same word will always have the same last letter)*
    *Symmetric –* $\forall(\alpha, \beta) \in R$ *implies* $(\beta,\alpha) \in R$ *if* $\alpha$ *ends with the same letter as* $\beta$*, then* $\beta$ *must end with the same letter as* $\alpha$
    *Not Antisymmetric -* $\forall(\alpha, \beta) \in R$ *and* $(\beta,\alpha) \in R$ *does not imply that* $\alpha = \beta$*; counter example: let* $\alpha$*= 'ae' and* $\beta$ *= 'e'*
    *Transitive- For any* $\alpha,\beta,\chi \in R$ *if* $(\alpha,\beta)$ *and* $(\beta,\chi)$ *hold then the last letter of* $\alpha$ *must be the same as the last letter of* $\beta$*, similarly the last letter of* $\beta$ *is the same as the last letter of* $\chi$*, therefore since equality is transitive this relation is transitive.*

    d. {(a,b) | every letter in a also appear in b}
    *Reflexive – because* $\forall \alpha \in R$ *implies* $(\alpha,\alpha) \in R$ *(*$\alpha$ *will always have the same letters of itself. The letters of* $\alpha$ *is always* $\alpha$ *subset of itself.*

*Not Symmetric: let $\alpha$ = dead, $\beta$ = deadbeef; Not Antisymmetric: counter example: let $\alpha$ = kleinberg, $\beta$ = rebelking so $(\beta,\alpha) \in R$ but $\alpha \neq \beta$*
*Transitive- Objective: For any $\alpha,\beta,\chi$ if $(\alpha,\beta) \in R$ and $(\beta,\chi) \in R$ then $(\beta,\chi) \in R$. Let A be the set formed by the letters in $\alpha$, and B be the set formed by the letters of $\beta$ and C be a set formed by the letters of $\chi$. Since $((A \subseteq B) \subseteq C)$ the relation must be transitive. The letters in $\chi$ will contain the letters in $\alpha$.*

2. R = {(a, a), (a,b), (b,c), (c,b), (c,c),(c,d)} is a relation on A = { a,b,c,d}.
   a. What is the symmetric closure of R?
   *Symmetric closure R is defined as such: if $(\alpha,\beta) \in R$ then $(\beta,\alpha) \in R$*

   *Construct a matrix A:*

| | *A* | *B* | *C* | *D* |
|---|---|---|---|---|
| *A* | *1* | *1* | *0* | *0* |
| *B* | *0* | *0* | *1* | *0* |
| *C* | *0* | *1* | *1* | *1* |
| *D* | *0* | *0* | *0* | *0* |

   *So when Symmetric closure is satisfied the matrix will be symmetric along the diagonal. We must add (b,a) and (d,c)*
   b. What is the transitive closure of R
   *$\forall \alpha,\beta,\chi \in R$ we need to check if $(\alpha,\beta) \in R$ and $(\beta,\chi) \in R$ then $(\alpha,\chi) \in R$*
   *(a,c), (b,b), (a,d), and (b,d)*
   c. What is the symmetric and transitive closure of R?

| | A | B | C | D |
|---|---|---|---|---|
| A | 1 | 1 | 1 | 1 |
| B | 1 | 1 | 1 | 1 |
| C | 1 | 1 | 1 | 1 |
| D | 1 | 1 | 1 | 1 |

   *Symmetric closure of symmetry and transitivity is a relation in which symmetry and transitivity hold among all elements. We therefore have to include the answers obtained from part a and part b. {(b,a), (d,c), (a,c), (b,b), (a,d), (b,d)} and then include the new symmetries and transitivites that are added to the set. {(c,a),(d,a),(d,b),(d,d)}. The union of these relations gives a full matrix.*
   d. What is the reflexive, symmetric closure of R?
   *Once again we have to include all the symmetric closures of part a: {(b,a) ,(d,c)}. We also have to include the diagonals: {(b,b) ,(d,d)}*

| | A | B | C | D |
|---|---|---|---|---|
| A | 1 | 1 | 0 | 0 |
| B | 1 | 1 | 1 | 0 |
| C | 0 | 1 | 1 | 1 |
| D | 0 | 0 | 1 | 1 |

PART B:

3. Consider the following Hasse diagram:

```
            A
          /   \
        b      c
      /   \   /  \
    d       e       f
     \     /  \     /
        g       h
```

   a. Which elements are maximal?
      *An element $\alpha$ is maximal if there exists no $\beta$ such that $\alpha < \beta$.*
      *Therefore a is the only maximal element: {a}*
   b. Which are minimal?
      *An element $\alpha$ is minimal if there exists no b such that b< $\alpha$. Therefore g and h are the only minimal elements. {g,h}*
   *c.* Find the upper bounds of {d,e}: *Upper Bound is defined as such: If $\upsilon \in S$ such that $\alpha <= \upsilon$ for all elements $\alpha \in S$ it is an upper bound. Therefore the upper bounds are a and b. {a, b}*
      >>Is there a least upper bound? *The least upper bound must be unique. It must be less than all the other upper bounds. Therefore b is a least upper bound. {b}*
   *d.* Find the lower bounds of {b,c}: *Lower Bounds is defined as such: If $\lambda \in S$ such that $\alpha >= \lambda$ for all elements $\alpha \in S$ it is a lower bound. Therefore {e,g,h} fit this criteria. Note f and d are not lower bounds because d there is no relation between c and d or f and b.*
      >>Is there a greatest lower bound? *The greatest lower bound must be unique. It must be greater than all other lower bounds. Therefore e is a greatest lower bound. {e}.*

4. Consider the following Hasse diagram. No node e. B connects to h and c connects to g.

```
            A
          /      \
        b          c
      /   \       /  \
    d       \   /      f
     \       /  \      /
         g        h
```

   a. Find the lower bounds of {b,c}. *{g,h}* Is there a greatest lower bound? *No*
   b. Find the upper bounds of {g,h}. *{b,c,a}* Is there a least upper bound? *No*
   c. Find the lower bounds of {d,c}. *{g}* Is there a greatest lower bound? *Yes, g.*
   d. Find the lower bounds of {a,b}. *{d,g,h,b}* Is there a greatest lower bound? *Yes, b.*

Part C:

5. Consider two relations R and S on some set A. Define two relations T and U on A.

   T = {(a,b) | (a,b) in R and (a,b) in S}

   U = {(a,b) | (a,b) in R or (a,b) in S }

   a. Suppose R and S are equivalence relations.
      - Is T an Equivalence Relation?
        - *Reflexive: $\forall \alpha$ in $\in T$ $(\alpha, \alpha) \in R$ and $(\alpha, \alpha) \in S$ because R and S are equivalence relations, therefore $(\alpha, \alpha) \in T$*
        - *Symmetry: Assume $(\alpha,\beta) \in T$ holds so $(\alpha,\beta) \in R$ and $(\alpha,\beta) \in S$ hold. Therefore by the definition of an equivalence relation $(\beta,\alpha) \in R$ and $(\beta,\alpha) \in S$. Hence $(\beta,\alpha) \in T$.*
        - *Transitive: Assume $(\alpha,\beta)$ and $(\beta,\chi) \in T$ . Therefore $(\alpha,\beta)$ and $(\beta,\chi) \in$ S and $(\alpha,\beta)$ and $(\beta,\chi) \in R$, because S and R are equivalence relations $(\alpha,\chi) \in$ S and $(\alpha,\chi) \in R$. Therefore $(\alpha,\chi) \in T$.*
        - *So T is an equivalence relation.*
      - Is U an equivalence Relation?
        - *Counter example to disprove transitivity: let R = $\{(\alpha,\beta),(\beta,\alpha),(\alpha,\alpha),(\beta,\beta)\}$ and S = $\{(\beta,\delta),(\delta,\delta), (\delta,\beta), (\beta,\beta)\}$. So U consists of $(\alpha,\beta)$ and $(\beta,\delta)$ but it doesn't have $(\alpha,\delta)$ therefore transitivity fails.*
        - *U is not an equivalence relation.*

   b. Suppose R and S are partially ordered
      - Is T a partial order?
        - *Reflexive: $\forall \alpha$ in $\in T$ $(\alpha, \alpha) \in R$ and $(\alpha, \alpha) \in S$ because R and S are equivalence relations, therefore $(\alpha, \alpha) \in T$*
        - *Antisymmetry: Assume $(\alpha,\beta)$ and $(\beta,\alpha) \in T$. Therefore $(\alpha,\beta) \in$ R and $(\alpha,\beta) \in S$. Hence, by the definition of a partial order $\alpha = \beta \in R$ and $\alpha = \beta \in S$. Therefore Antisymmetry holds for T.*
        - *Transitive: Assume $(\alpha,\beta)$ and $(\beta,\chi) \in T$ . Therefore $(\alpha,\beta)$ and $(\beta,\chi) \in$ S and $(\alpha,\beta)$ and $(\beta,\chi) \in R$, because S and R are equivalence relations $(\alpha,\chi) \in$ S and $(\alpha,\chi) \in R$. Therefore $(\alpha,\chi) \in T$.*
        - *So T is a partial order.*
      - Is U a partial order?
        - *Counter example to disprove transitivity: let R = $\{(\alpha,\beta),(\beta,\alpha), (\alpha,\alpha), (\beta,\beta)\}$ and S = $\{(\beta,\delta),(\delta,\delta), (\delta,\beta), (\beta,\beta)\}$. So U consists of $(\alpha,\beta)$ and $(\beta,\delta)$ but it doesn't have $(\alpha,\delta)$ therefore transitivity fails.*
        - *U is not a partial order.*

6. Suppose we build a simple graph on 10 vertices by flipping an unbiased coin to determine which edges are present. In other words, each possible edge e appears with probability ½.
    a. What is the expected number of edges for such a graph?
    *Total number of possible edges = (10 * 9)/2*
    *There are 10 vertices so 9 edges between them we must divide by two to eliminate directed edges.*
    *Let X (i) be the random variable associated with the $i^{th}$ edge. Where 0<=i<=45. Therefore E (x) = Sum (p (i)*X (i)) = (.5 * 1) * 45 = 22.*5

    *b.* What is the expected degree of a vertex?
    *We can analyze one vertex. Let X(i)  be the random variable associated to the ith edge coming into the vertex.. The probability of having an edge is .5 so E (x) = 9*(.5 * 1) = 4.5*