Computer Science 280
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Homework 3 Solutions
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Problem 1
a) What is the last digit in 7777^{7777} ?
$7777 = 7 \pmod{10}$
$7777^2 = 7^2 \pmod{10}$
…
$7777^{7777} = 7^{7777} \pmod{10}$
$7777^{7777} \bmod 10 = 7^{7777} \bmod 10$

$7^1 \bmod 10 = 7$
$7^2 \bmod 10 = 9$
$7^3 \bmod 10 = 3$
$7^4 \bmod 10 = 1$
$7^5 \bmod 10 = 7$ <-- notice that 7^5 is congruent to 7 modulo 10.
$7^6 \bmod 10 = 9$ <-- notice that 7^6 is congruent to 7^2 modulo 10

Using this observation, we can simplify $7^{7777} \bmod 10$
$7^{7777} \bmod 10 = 7^{(7777 \bmod 4)} \bmod 10$
$= 7^1 \bmod 10$
$= 7$

Last digit of 7777^{7777} is 7

b) $7777^{7777} \bmod 6$

$7777 = 1 \pmod 6$
$7777^2 = 1^2 \pmod 6$
$7777^3 = 1^3 \pmod 6$
…
$7777^{7777} = 1^{7777} \pmod 6$

$7777^{7777} \bmod 6 = 1^{7777} \bmod 6$
$= 1 \bmod 6$
$= 1$

Problem 2
a) $666^{666} \bmod 5$

$666 = 1 \pmod 5$
$666^2 = 1^2 \pmod 5$
$666^3 = 1^3 \pmod 5$
…
$666^{666} = 1^{666} \pmod 5$

$666^{666} = 1^{666} \pmod 5$
$666^{666} \bmod 5 = 1^{666} \bmod 5$
$= 1 \bmod 5$
$= 1$

b)
$765^{765} \bmod 8$

$765 = 5 \pmod 8$
$765^2 = 5^2 \pmod 8$
$765^3 = 5^3 \pmod 8$
…
$765^{765} = 5^{765} \pmod 8$

$765^{765} \bmod 8 = 5^{765} \bmod 8$
To find $5^{765} \bmod 8$, we can do the following

$5^1 \bmod 8 = 5$
$5^2 \bmod 8 = 1$
$5^3 \bmod 8 = 5$ <-- notice that 5^3 is congruent to 5 modulo 8
$5^4 \bmod 8 = 1$ <-- notice that 5^4 is congruent to 5^2 modulo 8

$5^{765} \bmod 8 = 5^{(765 \bmod 2)} \bmod 8$
$= 5^1 \bmod 8$
$= 5$

Therefore, $765^{765} \bmod 8 = 5$

Problem 3
The growth rate, from lowest to highest is as follows.

- $1 + \sin(n)$
- $\log(\log(n))$
- $\log n$
- $(\log n)^2$
- n
- $n \log n$
- $n^3 + n$
- 2^n

The order of 1+sin(n) is O(1) since the range of sine is [-1,1].
The order of $n^3 + n$ is $O(n^3)$ since the n^3 is the highest order polynomial term.

Problem 4
a)
I'll define a shorthand ONES(n) to be an integer with n digits of 1.
Find the gcd(ONES(75), ONES(165)).

Using Euclidean Algorithm
ONES(165) = ONES(75) * $(10^{90} + 10^{15})$ + ONES(15)
ONES(75) = ONES(15) * $(10^{60} + 10^{45} + 10^{30} + 10^{15})$
Hence, gcd(ONES(75), ONES(165)) = ONES(15), since ONES(15) is the last nonzero remainder

b)

Find the gcd($n^3 - n^2 + n - 1$, $n^3 - 1$).
Using Euclidean Algorithm

$n^3 - n^2 + n - 1 = (n^3 - 1)(1) + (-n^2 + n)$
$(n^3 - 1) = (-n^2 + n)(-n - 1) + (n - 1)$
$(-n^2 + n) = (n - 1)(-n)$

gcd($n^3 - n^2 + n - 1$, $n^3 - 1$) = $n - 1$, since $n - 1$ is the last nonzero remainder.

Problem 5

a)

Show that $5 \mid (144^{99} + 216^{99})$
This is the same as showing $(144^{99} + 216^{99}) = 0 \pmod 5$

$144 = 4 \pmod 5$ $144^2 = 4^2 \pmod 5$ $144^3 = 4^3 \pmod 5$ … $144^{99} = 4^{99} \pmod 5$	$216 = 1 \pmod 5$ $216^2 = 1^2 \pmod 5$ $216^3 = 1^3 \pmod 5$ … $216^{99} = 1^{99} \pmod 5$ $216^{99} = 1 \pmod 5$

Combining the two, we get
$(144^{99} + 216^{99}) = (4^{99} + 1) \pmod 5$
$(144^{99} + 216^{99}) = (4^{99} + 1) \pmod 5$ if and only if $(144^{99} + 216^{99}) \bmod 5 = (4^{99} + 1) \bmod 5$
Therefore, if $(4^{99} + 1) \bmod 5 = 0$ then $5 \mid (144^{99} + 216^{99})$

$4 = -1 \pmod 5$
$4^2 = 1 \pmod 5$
$4^3 = -1 \pmod 5$ <-- notice that 4^3 is congruent to 4 modulo 5.
$4^4 = 1 \pmod 5$
…
$4^{99} = -1 \pmod 5$
$(4^{99} + 1) \bmod 5 = (-1 + 1) \bmod 5$
$= 0 \bmod 5$
$= 0$

Therefore $5 \mid (144^{99} + 216^{99})$.

b)

Show that $5 \mid (n^5 + 4n)$ for any n.
We factor $(n^5 + 4n) = n(n^4 + 4)$
$5 \mid (n^5 + 4n)$ if and only if $5 \mid n$ or $5 \mid (n^4 + 4)$

if n is divisible by 5, then $5 \mid n(n^4 + 4)$
if n is not divisible by 5, then in order for $5 \mid (n^5 + 4n)$, $(n^4 + 4)$ must be divisible by 5

For all n where n is not divisible by 5.

If n is not divisible by 5, then n mod 5 will be in the set {1,2,3,4}
We will express this as

$n = \{1,2,3,4\} \pmod 5$
$n^2 = \{1^2,2^2,3^2,4^2\} \pmod 5 = \{1, 4, 4, 1\} \pmod 5$
$n^3 = \{1^3,2^3,3^3,4^3\} \pmod 5 = \{1, 3, 2, 4\} \pmod 5$
$n^4 = \{1^4,2^4,3^4,4^4\} \pmod 5 = \{1, 1, 1, 1\} \pmod 5$

$n^4 = 1 \pmod 5$
$4 = 4 \pmod 5$
$(n^4 + 4) = (1 + 4) \pmod 5$
$(n^4 + 4) \bmod 5 = 5 \bmod 5$
$(n^4 + 4) \bmod 5 = 0$
Therefore, $5 \mid (n^4 + 4)$ when n is not divisible by 5.
We have shown that $5 \mid (n^5 + 4n)$ for any n.

Problem 6
Show that $90 \mid 17^{11} + 163^{15}$

$90 = 2 * 5 * 9$

$17 = 1 \pmod 2$ $17^2 = 1^2 \pmod 2$ … $17^{11} = 1 \pmod 2$	$17 = 2 \pmod 5$ $17^2 = 2^2 \pmod 5$ … $17^{11} = 2^{11} \pmod 5$ $17^{11} = 3 \pmod 5$	$17 = 8 \pmod 9$ $17^2 = 8^2 \pmod 9$ … $17^{11} = 8^{11} \pmod 9$ $17^{11} = 8 \pmod 9$
$163 = 1 \pmod 2$ $163^2 = 1^2 \pmod 2$ … $163^{15} = 1^{15} \pmod 2$	$163 = 3 \pmod 5$ $163^2 = 3^2 \pmod 5$ … $163^{15} = 3^{15} \pmod 5$ $163^{15} = 2 \pmod 5$	$163 = 1 \pmod 9$ $163^2 = 1^2 \pmod 9$ … $163^{15} = 1^{15} \pmod 9$ $163^{15} = 1 \pmod 9$

$17^{11} + 163^{15} = (1+1) \pmod 2$
$17^{11} + 163^{15} = 2 \pmod 2$
$17^{11} + 163^{15} \bmod 2 = 0$

$17^{11} + 163^{15} = (3+2) \pmod 5$
$17^{11} + 163^{15} = 5 \pmod 5$
$17^{11} + 163^{15} \bmod 5 = 0$

$17^{11} + 163^{15} = (8+1) \pmod 9$
$17^{11} + 163^{15} = 9 \pmod 9$
$17^{11} + 163^{15} \bmod 9 = 0$

Since $17^{11} + 163^{15}$ is divisible by 2, 5, and 9, $90 \mid 17^{11} + 163^{15}$