Computer Science 280

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Homework 2 Solutions

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Part A

1. (a) "Some dog does not have his day."

   (b) "Some action has no equal and opposite reaction."

   (c) "Some golfer will never be defeated by a better golfer."

   (d) This one is a bit trickier. Let $P(x)$ mean that $x > 1$, and let $Q(x)$ mean that $x^2 > x$. So the statement that we want to negate is $\exists x (P(x) \rightarrow Q(x))$. By the definition of implication, this is equivalent to $\exists x (\neg P(x) \vee Q(x))$. When we negate this, we get $\forall x (\neg (\neg P(x) \vee Q(x)))$. Applying De Morgan's laws, we can simplify this to $\forall x (P(x) \wedge \neg Q(x))$. "For all x, $x > 1$ and $x^2 \leq x$" is therefore the correct negation.

2. We have the following two assumptions:

   (i) *Logic is difficult or not many students like logic.*

   (ii) *If mathematics is easy, then logic is not difficult.*

   Let's make a few definitions:

   - We will use l to represent "logic" and m to represent "mathematics".
   - Given $n \in \{l, m\}$, $D(n)$ will mean that the discipline n is difficult.
   - Given $n \in \{l, m\}$, $E(n)$ will mean that the discipline n is easy. Although it makes no difference in this problem, we should note that we do *not* assume that $(E(n) \Leftrightarrow \neg D(n))$.
   - Given $n \in \{l, m\}$, $S(n)$ will mean that many students enjoy discipline n. As one may expect, $\neg S(n)$ will mean that few students enjoy n.

   Now, we can symbolically express the two assumptions:

   (i) $(D(l) \vee \neg S(l))$

(ii) $(E(m) \rightarrow \neg D(l))$

Observe that (by the definition of implication and by contraposition, respectively) these two are equivalent to the following:

(i) $(S(l) \rightarrow D(l))$

(ii) $(D(l) \rightarrow \neg E(m))$

We are now ready to check the following claims:

(a) "Mathematics is not easy if many students like logic." This claim can be written as $(S(l) \rightarrow \neg E(m))$. Since we know from above that $(S(l) \rightarrow D(l))$, and that $(D(l) \rightarrow \neg E(m))$, we conclude that this claim is true.

(b) "Not many students like logic if mathematics is not easy." This claim can be written as $(\neg E(m) \rightarrow \neg S(l))$. By contraposition, we obtain $(S(l) \rightarrow E(m))$. Again, we know from our hypotheses that $(S(l) \rightarrow D(l))$, and that $(D(l) \rightarrow \neg E(m))$, so we conclude that this claim is false.

(c) "Mathematics is not easy or logic is difficult." We can write this claim as $(\neg E(m) \vee D(l))$, which is equivalent (by the definition of implication) to $(E(m) \rightarrow D(l))$. However, this claim contradicts $(E(m) \rightarrow \neg D(l))$, which is one of our hypotheses. Therefore the claim is false.

(d) "Logic is not difficult or mathematics is not easy." Symbolically, we express this claim as $(\neg D(l) \vee \neg E(m))$. If we convert this to an implication, we get $(D(l) \rightarrow \neg E(m))$, which is one of our hypotheses. The claim is therefore true.

(e) "If not many students like logic, then either mathematics is not easy or logic is not difficult." Here we have a complex claim: $(\neg S(l) \rightarrow (\neg E(m) \vee \neg D(l)))$. If we convert the right side into an implication, we get $(\neg S(l) \rightarrow (D(l) \rightarrow \neg E(m)))$. However, since $(D(l) \rightarrow \neg E(m))$ is one of our hypotheses, this claim becomes $(\neg S(l) \rightarrow \mathbf{T})$, which is equivalent to $(S(l) \vee \mathbf{T})$, which (by the domination law) is always true. Hence, the claim is true.

Alternate Solution: This question was ambiguous, so some may have interpreted the "either…or" phrase to imply "exclusive or". In this case, we have $(\neg S(l) \rightarrow (\neg E(m) \oplus \neg D(l)))$, which becomes $(S(l) \vee ((E(m) \wedge \neg D(l)) \vee (\neg E(m) \wedge D(l))))$, or $(S(l) \vee (E(m) \wedge \neg D(l)) \vee (\neg E(m) \wedge D(l)))$. Now suppose that not many students like logic, that mathematics is not easy, *and* that logic is not difficult. Symbolically, we represent this situation as $(\neg S(l) \wedge \neg E(m) \wedge \neg D(l))$. Notice that although the hypotheses are fulfilled here, the claim does not hold. The claim therefore does not follow from the hypotheses; it is false.

Part B

3. Let the universe of discourse for the variable x be the set of students in the class. For the variable y (representing year in school), let the universe of discourse be $\{f, s, j, s'\}$ ("freshman", "sophomore", "junior", and "senior" respectively). For the variable z (representing major), let the universe of discourse be $\{m, c\}$ ("mathematics" and "computer science" respectively). For brevity, we introduce the following notations:

   - $Y(x, y) \overset{\text{def}}{\Longleftrightarrow}$ x's year is y
   - $M(x, z) \overset{\text{def}}{\Longleftrightarrow}$ x's major is z

   Now we can write the statements symbolically:

   (a) "There is a student in the class who is a junior." We write this as $\exists x\,(Y(x, j))$, and this statement is true.

   (b) "Every student in the class is a computer science major." This can be written as $\forall x\,(M(x, c))$, and this statement is false.

   (c) "There is a student in the class who is neither a mathematics major nor a junior." We can write this as $\exists x\,(\neg M(x, m) \wedge \neg Y(x, j))$, and this statement is true.

   (d) "Every student in the class is either a sophomore or a computer science major." We write this as $\forall x\,(Y(x, s) \vee M(x, c))$, and this statement is false.

   (e) "There is a major such that there is a student in the class in every year of study with that major." This can be written as $\exists z \forall y \exists x\,(Y(x, y) \wedge M(x, z))$, and this statement is false.

4. We will first simplify the Boolean expressions, and then we will demonstrate equivalence through truth tables.

   (a)

$$
\begin{array}{clr}
 & p \rightarrow [p \wedge (q \vee \neg p)] & \\
\Longleftrightarrow & p \rightarrow [(p \wedge q) \vee (p \wedge \neg p)] & \text{(distributive law)} \\
\Longleftrightarrow & p \rightarrow [(p \wedge q) \vee \mathbf{F}] & \text{(since } (p \wedge \neg p) \Leftrightarrow \mathbf{F}) \\
\Longleftrightarrow & p \rightarrow [p \wedge q] & \text{(identity law)} \\
\Longleftrightarrow & \neg p \vee [p \wedge q] & \text{(definition of implication)} \\
\Longleftrightarrow & [\neg p \vee p] \wedge [\neg p \vee q] & \text{(distributive law)} \\
\Longleftrightarrow & \mathbf{T} \wedge [\neg p \vee q] & \text{(since } (\neg p \vee p) \Leftrightarrow \mathbf{T}) \\
\Longleftrightarrow & (\neg p \vee q) & \text{(identity law)} \\
\Longleftrightarrow & (p \rightarrow q) & \text{(definition of implication)}
\end{array}
$$

   The following truth table demonstrates the equivalence of the initial and final expressions:

$(p$	$\rightarrow$	$[p$	$\wedge$	$(q$	$\vee$	$\neg$	$p)])$	$\leftrightarrow$	$(p$	$\rightarrow$	$q)$
T	T	T	T	T	T	F	T	**T**	T	T	T
T	F	T	F	F	F	F	T	**T**	T	F	F
F	T	F	F	T	T	T	F	**T**	F	T	T
F	T	F	F	F	T	T	F	**T**	F	T	F

   (b)

$$
\underbrace{\left[\overbrace{(p \wedge r)}^{\substack{\text{False if} \\ p \text{ is false}}} \rightarrow \overbrace{(p \vee q)}^{\substack{\text{True if} \\ p \text{ is true}}}\right]}_{\text{Always true}} \rightarrow (r \rightarrow q)
$$

$$
\begin{array}{clr}
\Longleftrightarrow & \mathbf{T} \rightarrow (r \rightarrow q) & \text{(left side is always true)} \\
\Longleftrightarrow & \neg \mathbf{T} \vee (r \rightarrow q) & \text{(definition of implication)} \\
\Longleftrightarrow & \mathbf{F} \vee (r \rightarrow q) & (\neg \mathbf{T} \Leftrightarrow \mathbf{F}) \\
\Longleftrightarrow & (r \rightarrow q) & \text{(identity law)}
\end{array}
$$

   This truth table verifies the equivalence of the two expressions:

$([(p$	$\wedge$	$r)$	$\rightarrow$	$(p$	$\vee$	$q)]$	$\rightarrow$	$(r$	$\rightarrow$	$q))$	$\leftrightarrow$	$(r$	$\rightarrow$	$q)$
T	T	T	T	T	T	T	T	T	T	T	**T**	T	T	T
T	T	T	T	T	T	F	F	T	F	F	**T**	T	F	F
T	F	F	T	T	T	T	T	F	T	T	**T**	F	T	T
T	F	F	T	T	T	F	T	F	T	F	**T**	F	T	F
F	F	T	T	F	T	T	T	T	T	T	**T**	T	T	T
F	F	T	T	F	F	F	F	T	F	F	**T**	T	F	F
F	F	F	T	F	T	T	T	F	T	T	**T**	F	T	T
F	F	F	T	F	F	F	T	F	T	F	**T**	F	T	F

Part C

5. We will represent each statement symbolically in checking the validity of the claims.

   (a) We have the following hypotheses:

      (i) *If there is gas in the car then I will go to K-Mart.* We will use p to signify that there is gas in the car, and q to signify that I go to K-Mart. Therefore, we have $(p \rightarrow q)$.

      (ii) *If I go to K-Mart then I will buy some Martha Stewart designer shower curtains.* We will use r to signify that I buy the shower curtains. Therefore, we have $(q \rightarrow r)$.

      (iii) *I do not buy shower curtains.* We have $\neg r$.

   We want to check the following conclusion: "There is no gas in the car or the car is broken." We will use s to signify that the car is broken. The conclusion can therefore be written symbolically as $(\neg p \vee s)$.

   By contraposition, we can rewrite (i) and (ii):

      (i) $(\neg q \rightarrow \neg p)$

      (ii) $(\neg r \rightarrow \neg q)$

   Since we know by (iii) that $\neg r$ holds, it is clear from (ii) that $\neg q$ must also hold, and from (i) it is clear that $\neg p$ therefore holds as well. This is sufficient (by the addition rule) to guarantee that $(\neg p \vee s)$ is true, so the conclusion is valid.

   (b) We are given the following hypotheses:

(i) *Everyone in the Discrete Structures class loves proofs.* Let the universe of discourse for the variable x be the set of people in the Discrete Structures class, and let $P(x)$ signify that person x loves proofs. We therefore have $\forall x\,(P(x))$.

(ii) *Someone in the Discrete Structures class has never taken History.* Let $H(x)$ signify that person x has taken History. We therefore have $\exists x\,(\neg H(x))$.

We want to check the following conclusion: "Someone who loves proofs has never taken History." Symbolically, we write this as $\exists x\,(\neg H(x) \wedge P(x))$. By (ii), we know (using existential instantiation) that for some person y in the class, $\neg H(y)$ is true. By (i), we also know (using universal instantiation), that $P(y)$ is true. By conjunction, we now have $(\neg H(y) \wedge P(y))$, and since y is from the universe of discourse of the variable x, this proves that the claim is true.

6. We want to find a closed formula for the sequence $\{a_n\}$, defined recursively below:

- $a_1 \stackrel{\text{def}}{=} 3$
- $a_2 \stackrel{\text{def}}{=} 5$
- $a_{n+1} \stackrel{\text{def}}{=} 3a_n - 2a_{n-1}$

First, observe that for large values of n (we want $n \geq 4$ to prevent problems with indices),

$$\begin{aligned}
a_{n+1} - a_n &= 2\,(a_n - a_{n-1}) \\
&= 2^2\,(a_{n-1} - a_{n-2}) \\
&= 2^3\,(a_{n-2} - a_{n-3}) \\
&\ \vdots \quad \vdots \\
&= 2^{n-1}\,(a_2 - a_1) \\
&= 2^{n-1}\,(5 - 3) \\
&= 2^{n-1}\,(2) \\
&= 2^n.
\end{aligned}$$

We would thus expect the formula to be of the form $a_n = 2^n + m$, where m is some integer. Given the values of a_1 and a_2, it only makes sense to choose $m = 1$. By induction on n, we will now verify that $a_n = 2^n + 1$.

Claim: $a_n = 2^n + 1$.

Basis: We have two base cases here ($n = 1$ and $n = 2$):

$$\begin{aligned} a_1 &\overset{\text{def}}{=} 3 \\ &= 2^1 + 1 \quad \text{(confirmed for } n = 1\text{)} \end{aligned}$$

$$\begin{aligned} a_2 &\overset{\text{def}}{=} 5 \\ &= 2^2 + 1 \quad \text{(confirmed for } n = 2\text{)} \end{aligned}$$

Induction Hypothesis: There exists a natural number k for which $1 \leq n \leq k$ implies that $a_n = 2^n + 1$.

Induction Step: Consider a_{k+1}. Remember that the key is to use what you know in proving your claim (and *not* to merely show that your claim is consistent with what you know):

$$\begin{aligned} a_{k+1} &\overset{\text{def}}{=} 3a_k - 2a_{k-1} \\ &= 3\left(2^k + 1\right) - 2\left(2^{k-1} + 1\right) \quad \text{(by induction hypothesis)} \\ &= 3\left(2^k\right) + 3 - 2^k - 2 \\ &= (3 - 1)\left(2^k\right) + 3 - 2 \\ &= 2\left(2^k\right) + 1 \\ &= 2^{k+1} + 1 \quad \text{(confirmed for } n = k + 1\text{)} \end{aligned}$$

We can therefore conclude that $a_n = 2^n + 1$ for all natural numbers $n \geq 1$.