CS280 HW1 Solution Set
Spring2002

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1-)

a-) $a_1 = 4*1-2 = 2$

$$a_{n+1} = 4(n+1)-2 = 4n-2+4 = a_n + 4$$

b-)

$$a_1 = 1+(-1)^1 = 0$$

$$a_{n+1} = 1+(-1)^{n+1} = ((1+(-1)^n)-1)*(-1)+1 = -1(a_n - 1)+1 = 2-a_n$$

c-) $a_{n+1} = (n+1)(n+2) = n(n+1) + 2(n+1) = a_n + 2(n+1)$

now, we need to get rid of the n term. We know that:

$$a_{n-1} = (n-1)n$$

$$a_n - a_{n-1} = n(n+1) - n(n-1) = 2n$$

$$a_n - a_{n-1} + 2 = 2n + 2 = 2(n+1)$$

substitute into the recursive expression:

$$a_{n+1} = a_n + (a_n - a_{n-1} + 2)$$

$$a_{n+1} = 2a_n - a_{n-1} + 2$$

so the solution is:

$$a_1 = 1*2 = 2$$

$$a_2 = 2*3 = 6$$

$$a_{n+1} = 2a_n - a_{n-1} + 2$$

d-) Two possible solutions I can think of:

solution 1:

$$a_1 = 1$$

$$a_{n+1} = (n+1)^2 = n^2 + 2n + 1 = a_n + 2\sqrt{a_n} + 1$$

solution 2:

by similar reasoning(actually its exactly the same!!) to c-), we get:

$a_1 = 1$

$a_2 = 4$

$a_{n+1} = 2a_n - a_{n-1} + 2$

2-)

We want to prove inductively that:
$\forall n$ (The plane is divided by n lines) $\Rightarrow$ (the regions can be colored using blue and red such that no two adjacent regions have the same color)

Note that two regions are adjacent if and only if they share an edge (I will use the word edge instead of side here to avoid ambiguity)

Basis: The plane is divided by 0 lines =>we can color the only region blue=> we can color it using blue and red such that no two adjacent regions have the same color

Inductive Hypothesis: (The plane is divided by n lines) $\Rightarrow$ (the regions can be colored using blue and red such that no two adjacent regions have the same color)

Induction: Given any n+1 lines dividing the plane, we can remove a line in the plane to make a total of n lines in the plane. Therefore the arrangement of n+1 lines can be formed as such from an arrangement of n lines by adding the desired line to make an arrangement of n+1 lines in the plane.

Let us start with our initial n lines. By the Induction Hypothesis, we can color the regions such that no two adjacent regions have the same color. Color them to achieve this. No two adjacent regions have the same color if and only if all edges obey predicate P:

> P(s): At any point on edge s, the color to that point's right is different from the color to that point's left

Now, add the desired line, let's call it line λ. Invert all the colors on the right of that line (blue becomes red and red becomes blue).

For all edges not on λ, we know that prior to the inversion all such edges obeyed predicate P. After the inversion, each edge to the right of λ had the colors on both its right and its left inverted, which implies that the colors remained different!

(Blue-Red became Red-Blue). Edges to the left of λ were not affected, and still obey predicate P. Therefore:

Even after adding line λ and inverting colors to λ's right, all edges NOT on λ still obey predicate P.

Now, what about edges ON line λ?

When we added line λ, and before inverting colors, each edge on λ passed through what was a single region before λ came into being (by the definition of an edge). Therefore each edge on λ had the same colors to both it's left and it's right. Then came our inversion, which inverted all colors to the right of λ, causing each edge on λ to have different colors to the left and right.

Therefore, after adding λ and inverting the right side of λ:

All edges on λ have different colors to the left and right

Thus we conclude that after adding λ and inverting colors to λ's right:

All edges, be they on λ or not, have different colors to their right and left

Hence, given the induction hypothesis, we have proved that an arbitrary arrangement of n+1 lines can be colored such that no two adjacent regions have the same color:

(The plane is divided by n+1 lines) $\Rightarrow$ (the regions can be colored using blue and red such that no two adjacent regions have the same color)

Conclusion:

Given the basis case and the induction, we conclude that:

$\forall n$ (The plane is divided by n lines) $\Rightarrow$ (the regions can be colored using blue and red such that no two adjacent regions have the same color)

3-)

Basis:

0 lines in the plane $\Rightarrow$ plane is divided into 1 region. $\Rightarrow$ plane is divided into 1=1+0(0+1)/2 regions

Inductive Hypothesis:

n lines in the plane $\Rightarrow$ plane is divided into 1+n(n+1)/2 regions

Induction:

In an arrangement of n+1 lines, we can remove a line to get an arrangement of n lines. Therefore, the arrangement of n+1 lines can be formed from an arrangement of n lines by adding the desired line.

According to the inductive hypothesis, before adding the extra line the plane has 1+n(n+1)/2 regions.

Now, we add the desired line λ. Since the line is infinite (not a segment) and is not parallel to any other line (according to the question specification), it must intersect each of the n lines already in the plane. Since the question also specifies that no 3 lines can intersect at a point, we know that our new line λ will intersect each of the n lines in the plane separately.

Since λ will intersect n lines separately, these n lines will chop λ up into n+1 segments (this is easy to see intuitively, but can be proved inductively). Each of the n+1 segments passes through what was one region before λ was added. Hence, n+1 regions are divided into 2 each, generating n+1 more regions. So after adding λ we end up with this many regions:

1+n(n+1)/2 +n+1 =1+(n+1)(n+2)/2

Therefore, if our inductive hypothesis is true then:
n+1 lines in the plane $\Rightarrow$ plane is divided into 1+(n+1)(n+2)/2 regions

Or, more precisely:
IH (n) $\Rightarrow$ IH (n+1)

Where IH is the inductive hypothesis predicate applied to variable n.

Conclusion:

$\forall n$ (n lines in the plane $\Rightarrow$ plane is divided into 1+n(n+1)/2 regions)

4-)

The statement is false. Proof by counter-example:

Take n = 41
$n^2 + n + 41 = 41^2 + 41 + 41 = 41(41+1+1) = 41 * 43$

41*43 is obviously divisible by 41, which is not itself nor 1. This means that it is not prime.

Therefore it is not true that $n^2 + n + 41$ is prime for all natural numbers n.

5-)

We want to prove inductively that:

$$\forall n \forall x_1, x_2 ... x_n \left(\left| \sum_{k=1}^{n} x_k \right| \le \sum_{k=1}^{n} |x_k| \right)$$

Basis:

For n=0, the absolute value of the sum of 0 numbers =0. The sum of the absolute values of 0 numbers also=0. They are equal. Therefore the inductive hypothesis below is true for n=0.

Inductive Hypothesis:

$$\forall x_1, x_2 ... x_n \left(\left| \sum_{k=1}^{n} x_k \right| \le \sum_{k=1}^{n} |x_k| \right)$$

Induction:

For any $x_1, x_2 ... x_{n+1}$:

$$\left| \sum_{k=1}^{n+1} x_k \right| = \left| \sum_{k=1}^{n} x_k + x_{n+1} \right|$$

now, if $\sum_{k=1}^{n} x_k$ and x_{n+1} have the same sign(+ve or -ve) then the above expression is equal to:

$$\left| \sum_{k=1}^{n} x_k \right| + |x_{n+1}|$$

otherwise, if they have different signs, then the expression is equal to the difference between their absolute values:

$$\left| \sum_{k=1}^{n} x_k \right| - |x_{n+1}|$$

Therefore, the greatest value $\left| \sum_{k=1}^{n+1} x_k \right|$ can have is:

$$\left| \sum_{k=1}^{n} x_k \right| + |x_{n+1}|$$

and it follows that:

IEQ1: $\left| \sum_{k=1}^{n+1} x_k \right| \le \left| \sum_{k=1}^{n} x_k \right| + |x_{n+1}|$

Using our induction hypothesis:

IH: $\left| \sum_{k=1}^{n} x_k \right| \le \sum_{k=1}^{n} |x_k|$

Combine IH and IEQ1 to get:

$$\left|\sum_{k=1}^{n+1} x_k\right| \le \left|\sum_{k=1}^{n} x_k\right| + |x_{n+1}| \le \sum_{k=1}^{n} |x_k| + |x_{n+1}|$$
$$\left|\sum_{k=1}^{n+1} x_k\right| \le \sum_{k=1}^{n+1} |x_k|$$

So, given our IH (n) predicate, we proved that:

$$\forall x_1, x_2 ... x_{n+1}\left(\left|\sum_{k=1}^{n+1} x_k\right| \le \sum_{k=1}^{n+1} |x_k|\right)$$

Which is IH (n+1)

<u>Conclusion:</u>

$$\forall n \forall x_1, x_2 ... x_n\left(\left|\sum_{k=1}^{n} x_k\right| \le \sum_{k=1}^{n} |x_k|\right)$$

6-)
The inductive by itself step is flawless. The basis step by itself is also flawless. However, it is not correct to combine them to form an inductive proof.

The reason is that the inductive step uses a^n and a^{n-1} to prove the inductive hypothesis for n+1. Therefore, any valid Basis step MUST define the first two elements of the sequence. Only by defining the first two elements (a^0 and a^1), and proving the inductive hypothesis for these elements can we begin to construct a valid inductive proof. If we modify the Basis step and define a^0 and a^1, we would realize that the Inductive Hypothesis actually DOES NOT hold for the Basis case, hence destroying the inductive argument before it even begins.