CS280 Homework 8 Solutions

Section 4.6

10. b) This is a stars and bars problem with 36 stars and 6 slots (or 5 bars), so the number of possibilities is

$$C(36+6-1,36) = \frac{41!}{36!5!} = 749398$$

d) We can break the problem into the case of no broccoli, 1 broccoli, and 2 broccoli and use stars and bars in each case, with broccoli excluded from further counting (so only 5 slots):

$$C(24+5-1,24) + C(23+5-1,23) + C(22+5-1,22) = 52975$$

e) We can set aside 5 chocolates and 3 almonds, then do stars and bars with 24-8=16 stars and 6 slots:

$$C(16+6-1,16) = 20349$$

28. Think of AAA as one unit and each remaining letter as one unit, so there are 6 units total, one of which is repeated. Hence there are $6! = 720$ possible permutations, then we divide by 2! to account for the double counting coming from the repeated unit. This gives 360 strings that can be formed.

36. We want to make permutations of the 4 possible moves, or in other words make permutations of $4+3+5+4=16$ total objects with 4 of the first type, 3 of the second, 5 of the third, and 4 of the fourth. Hence the number of possibilities is

$$\frac{16!}{4!3!5!4!} = 50450400$$

40. a) This is a stars and bars problem with 12 stars and 4 slots, so the number of ways is $C(12+4-1,12) = 455$.

b) The number of ways can be found by taking each of the possibilities in part (a), then using the fact that for each of these distributions, each of the permutations of the twelve books gives a different arrangement on the shelves. Hence there are $455(12!) = 2.17945 \cdot 10^{11}$ possible arrangements.

Section 5.1

8. a) The initial condition is $a_0 = 6 \cdot 10^9$. The recurrence relation is $a_n = 1.013a_{n-1}$ for $n > 0$.

b) Iterating the recurrence relation as in example 3 of section 5.1, we see that $a_n = (1.013)^n a_0 = (1.013)^n 6 \cdot 10^9$.

c) The question is asking for the value of a_{21}, which from part (b) is $(1.013)^{21} 6 \cdot 10^9 = 7.869 \cdot 10^9$.

24. a) Consider the case of a string of length n that ends in a 1. If this is a valid string (has two consecutive 0's), then the first $n-1$ digits must be valid. Hence there are a_{n-1} possibilities. Likewise, if the original string ends in a 2, there are a_{n-1} possibilities. If the original string ends in a 0, we consider 3 cases. If the $n-1$st digit is a 1, then the first $n-2$ digits must be valid, so there are a_{n-2} possibilities. Likewise if the $n-1$st digit is a 2, there are a_{n-2} possibilities. If the $n-1$st digit is a 0, then the last two digits are 0, so the first $n-2$ can be anything, so there are 3^{n-2} possibilities. Adding all of these terms, we get $a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$.

b) Since a string with 1 digit can't have two consecutive 0's, $a_1 = 0$. For strings with 2 digits, there is exactly one way to have two consecutive 0's, so $a_2 = 1$. Together, these two conditions give the initial conditions for the recurrence relation.

c) Using the formula in part (a) with the initial conditions in (b) and iterating, we have

$$a_3 = 3^1 + 2(1) + 2(0) = 5$$

$$a_4 = 3^2 + 2(5) + 2(1) = 21$$
$$a_5 = 3^3 + 2(21) + 2(5) = 79$$
$$a_6 = 3^4 + 2(79) + 2(21) = 281$$

Hence the number of ternary strings of length 6 with two consecutive 0's is 281.

44. Comparing the values of $J(n)$ with the values of n, we see that $J(n)$ is a repeating sequence of the positive odd integers, resetting to 1 each time n is exactly a power of two, then increasing by 2 for each integer until the next power of 2. In symbols, if $n = 2^m + k$, then $J(n)$ doesn't depend on m but is exactly $2k + 1$. But we can write m as $\lfloor \log_2 n \rfloor$, and then $k = n - 2^m$. So the formula for J is

$$J(n) = 2(n - 2^{\lfloor \log_2 n \rfloor}) + 1$$

46. For this induction, the base case has two parts: $J(1) = 1$, which holds since $\log_2 1 = 0$ and $2(1 - 2^0) + 1 = 1$, and $J(2) = 1$, which holds since $\log_2 2 = 1$ and $2(2 - 2^1) + 1 = 1$. For the induction hypothesis, we use the extended form of induction and assume for some $n \geq 2$ and $s = 1, \ldots, n$ that

$$J(s) = 2(s - 2^{\lfloor \log_2 s \rfloor}) + 1,$$

and will prove that

$$J(n+1) = 2(n + 1 - 2^{\lfloor \log_2 (n+1) \rfloor}) + 1.$$

If $n+1$ is even, say $n+1 = 2k$, then from problem 45 we know that $J(n+1) = J(2k) = 2J(k) - 1$. By the induction hypothesis,

$$\begin{aligned} J(k) &= 2(k - 2^{\lfloor \log_2 k \rfloor}) + 1 \\ &= 2k - 2^{\lfloor \log_2 k \rfloor + 1} + 1 \\ &= n + 1 - 2^{\lfloor \log_2 (n+1) \rfloor} + 1, \end{aligned}$$

since $\lfloor \log_2 k \rfloor + 1 = \lfloor \log_2 2k \rfloor$. Using this formula for $J(k)$ in $2J(k) - 1$, we obtain

$$J(n+1) = 2(n + 1 - 2^{\lfloor \log_2 n+1 \rfloor}) + 2 - 1,$$

which simplifies to the desired form.

If $n + 1$ is odd, say $n + 1 = 2k + 1$, then from problem 45 we know that $J(n + 1) = J(2k + 1) = 2J(k) + 1$. As before, after some simplification, the induction hypothesis gives $J(k) = 2k + 1 - 2^{\lfloor \log_2 k \rfloor + 1}$, which simplifies to $n + 1 - 2^{\lfloor \log_2 (n+1) \rfloor}$ by using the fact that $n + 1 = 2k + 1$ and $\lfloor \log_2(2k + 1) \rfloor = \lfloor \log_2(k + 1/2) \rfloor + 1$, and the fact that adding 1/2 to an integer k can't increase $\log_2$ to the next integer. Putting this all together, we get $J(n + 1) = 2(n + 1 - 2^{\lfloor \log_2 (n+1) \rfloor}) + 1$, as desired.

Section 5.5

6. a) Since $A_1 \subseteq A_2 \subseteq A_3$, we have $|A_1 \cup A_2 \cup A_3| = |A_3| = 10000$.

b) Since the three sets are disjoint, the inclusion-exclusion principle simplifies to $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| = 11100$.

c) Using the inclusion-exclusion formula, we have $|A_1 \cup A_2 \cup A_3| = 100 + 1000 + 10000 - (2 + 2 + 2) + 1 = 11095$.

24. Let A_1 be the event of exactly 3 tails, A_2 be the event of the first and last tosses being tails, and A_3 be the event of the second and fourth tosses being heads. Then $|A_1| = C(5, 3)$, $|A_2| = 2^3$, $|A_3| = 2^3$, $|A_1 \cap A_2| = C(3, 1)$, $|A_1 \cap A_3| = 1$, $|A_2 \cap A_3| = 2$, and $|A_1 \cap A_2 \cap A_3| = 1$. Hence $|A_1 \cup A_2 \cup A_3| = 10 + 8 + 8 - (3 + 1 + 2) + 1 = 21$. Since there are 2^5 total possible sequences, the probability of at least one of the events is 21/32.

Section 5.6

6. Let P_i be the property "divisible by i^2". Then using the alternative form of the inclusion-exclusion principle, and using the integers 1 to 99 as the set of interest,

$$N(P_2'P_3'P_4') = 99 - (N(P_2) + N(P_3) + N(P_4)) + (N(P_2P_3) + N(P_2P_4) + N(P_3P_4)) - N(P_2P_3P_4).$$

Using $N(P_i) = \lfloor 99/i^2 \rfloor$, we get

$$N(P_2'P_3'P_4') = 99 - (24 + 11 + 6) + (2 + 6 + 0) - 0 = 66.$$

However, there remain the squares 25, 36, 49, 64, and 81. Of these, 36, 64, and 81 are divisible by 4 or 9, so have already been excluded. So we must also exclude 25, 50, 75, 49, and 98, to get a total of $64 - 5 = 61$.

26. Any permutation ending in 1, 2, 3 must start with 4, 5, 6 (in some order), and any such permutation is a derangement. Hence the number of such derangements is the number of permutations of $\{1, 2, 3\}$ times the number of permutations of $\{4, 5, 6\}$, which is 3! times 3! or 36.