

CS 280 Solutions – Assignment 7

Section 4.4

14. Number of different hands of 5 cards from deck of 52
= $C(52, 5)$
= 2 598 960
Number of five-card poker hands that contain cards of 5 different kinds
= Number of ways to pick 5 kinds x Number of ways to pick 1 card each kind
= $C(13, 5) \times (C(4, 1))^5$
= 1 317 888
Probability of five-card poker hand containing cards of 5 different kinds
= $1\,317\,888 / 2\,598\,960 (\approx 0.5071)$
18. Number of different hands of 5 cards from deck of 52 = 2 598 960
Number of straight flushes
= (# ways to pick a suit) x (# ways to pick 5 consecutive kinds in a suit)
= $C(4, 1) \times 10$
= 40
Probability of straight flush
= $40 / 2\,598\,960$
= $1 / 64\,974 (\approx 1.539 \times 10^{-5})$
- 26(a) Total number of ways to choose 6 integers out of 40
= $C(40, 6)$
Total number of ways to choose 6 integers out of 34 (incorrect ones)
= $C(34, 6)$
Probability of selecting none of the correct 6 integers
= $C(34, 6) / C(40, 6)$
= $1\,344\,904 / 3\,838\,380 (\approx 0.3504)$
- 26(b) Same as (a):
Probability of selecting none of the correct 6 integers
= $C(42, 6) / C(48, 6)$
= $5\,245\,786 / 12\,271\,512$
 ≈ 0.4275
- 34(a) $p(E_1) = p(E_2) = 0.5$
 $p(E_1 \cap E_2) = p(E_1)p(E_2) = 0.25$
Yes, E_1 and E_2 are independent.
- Let $A_1A_2A_3$ denote the sequence of outcomes for 3 fair tosses of a coin.
 $A_{1,2,3} = \{H,T\}$

34(b) $p(E_1) = 0.5$
 $p(E_2) = p(\text{HHT}) + p(\text{THH})$
 $= 2 / 8 = 0.25$
 $p(E_1 \cap E_2) = p(\text{THH}) = 1 / 8 = 0.125$
 $p(E_1)p(E_2) = 0.5 \times 0.25 = 0.125$
 Therefore, $p(E_1 \cap E_2) = p(E_1)p(E_2)$
 Yes, E_1 and E_2 are independent.

34(c) $p(E_1) = 0.5$
 $p(E_2) = p(\text{HHT}) + p(\text{THH})$
 $= 2 / 8 = 0.25$
 $p(E_1 \cap E_2) = p(\emptyset) = 0$
 $p(E_1)p(E_2) = 0.5 \times 0.25 = 0.125$
 Therefore, $p(E_1 \cap E_2) \neq p(E_1)p(E_2)$
 No, E_1 and E_2 are not independent.

Note: In these questions, we implicitly assume that $p(H) = p(T) = 0.5$

Section 4.5

- 8 Let $P(n)$ be the proposition that
 $p(E_1 \cap E_2 \cap \dots \cap E_n) \geq p(E_1) + p(E_2) + \dots + p(E_n) - (n - 1)$

Base case:

For $n = 1$

LHS = $p(E_1)$

RHS = $p(E_1) - (1 - 1) = p(E_1)$

Trivially, $p(E_1) \geq p(E_1)$

Thus, $P(n)$ holds true for $n = 1$

For $n = 2$

LHS = $p(E_1 \cap E_2)$

RHS = $p(E_1) + p(E_2) - (2 - 1) = p(E_1) + p(E_2) - 1$

By Inclusion-exclusion principle,

$p(E_1 \cap E_2) = p(E_1) + p(E_2) - p(E_1 \cup E_2)$

Since $p(E_1 \cup E_2) \leq 1$,

$p(E_1 \cap E_2) \geq p(E_1) + p(E_2) - 1$

Thus, $P(n)$ holds true for $n = 2$

Induction hypothesis:

Assume that $P(k)$ holds true for some $k > 0$, i.e.

$p(E_1 \cap E_2 \cap \dots \cap E_k) \geq p(E_1) + p(E_2) + \dots + p(E_k) - (k - 1)$

Inductive step:

Consider $P(k+1)$:

$p(E_1 \cap E_2 \cap \dots \cap E_k \cap E_{k+1}) = p((E_1 \cap E_2 \cap \dots \cap E_k) \cap E_{k+1})$

By Inclusion-exclusion principle,

$$\begin{aligned} & p((E_1 \cap E_2 \cap \dots \cap E_k) \cap E_{k+1}) \\ &= p(E_1 \cap E_2 \cap \dots \cap E_k) + p(E_{k+1}) - p((E_1 \cap E_2 \cap \dots \cap E_k) \cup E_{k+1}) \end{aligned}$$

Since $p((E_1 \cap E_2 \cap \dots \cap E_k) \cup E_{k+1}) \leq 1$,

$$p((E_1 \cap E_2 \cap \dots \cap E_k) \cap E_{k+1}) \geq p(E_1 \cap E_2 \cap \dots \cap E_k) + p(E_{k+1}) - 1$$

By induction hypothesis, $p(E_1 \cap E_2 \cap \dots \cap E_k) \geq p(E_1) + p(E_2) + \dots + p(E_k) - (k - 1)$

Therefore,

$$\begin{aligned} & p((E_1 \cap E_2 \cap \dots \cap E_k) \cap E_{k+1}) \\ & \geq p(E_1 \cap E_2 \cap \dots \cap E_k) + p(E_{k+1}) - 1 \\ & \geq p(E_1) + p(E_2) + \dots + p(E_k) - (k - 1) + p(E_{k+1}) - 1 \\ & = p(E_1) + p(E_2) + \dots + p(E_{k+1}) - ((k+1) - 1) \end{aligned}$$

Thus, $P(k) \rightarrow P(k+1)$

Since $P(n)$ is true for the base cases $P(1)$ and $P(2)$,
by induction, $P(n)$ is true for all $n > 0$.

$$\begin{aligned} 18 \quad S &= \{000, 001, 010, 011, 100, 101, 110, 111\} \\ E &= \{100, 010, 001, 111\} \\ F &= \{100, 101, 110, 111\} \end{aligned}$$

$$\begin{aligned} E \cap F &= \{100, 111\} \\ p(E) &= 4 / 8 = 0.5 \\ p(F) &= 4 / 8 = 0.5 \\ p(E)p(F) &= (0.5)^2 = 0.25 \\ p(E \cap F) &= 2 / 8 = 0.25 \end{aligned}$$

Thus, $p(E \cap F) = p(E)p(F)$

Therefore, E and F are independent.

$$22(a) \quad P(\text{"0"}) = P(\text{"1"}) = 0.5$$

$$\begin{aligned} & \text{Probability that a randomly generated bit string does not contain a 0} \\ &= \text{Probability that each character in string is 1} \\ &= (0.5)^{10} \\ &= 1 / 1024 (\approx 9.766 \times 10^{-4}) \end{aligned}$$

$$22(b) \quad P(\text{"1"}) = 0.6; \quad P(\text{"0"}) = 1 - P(\text{"1"}) = 0.4$$

$$\begin{aligned} & \text{Probability that a randomly generated bit string does not contain a 0} \\ &= \text{Probability that each character in string is 1} \\ &= (0.6)^{10} \\ &= 59\,049 / 9\,765\,625 (\approx 0.006\,047) \end{aligned}$$

$$22(c) P(\text{"1"}) = (0.5)^i$$

Probability that a randomly generated bit string does not contain a 0

= Probability that *each* character in string is 1

$$= (0.5)^1(0.5)^2(0.5)^3 \dots (0.5)^{10}$$

$$= 0.5^{(1+2+3+\dots+10)}$$

$$= 0.5^{55}$$

$$= 1/2^{55} (\approx 2.776 \times 10^{-17})$$

26(b) Probability of at least one success

= 1 – Probability of no success

$$= 1 - (1 - p)^n$$

26(d) Probability of at least two successes

= 1 – Probability of no success – Probability of 1 success

$$= 1 - (1 - p)^n - C(n, 1)p(1 - p)^{n-1}$$

$$= 1 - (1 - p)^n - np(1 - p)^{n-1}$$

If $n < 2$, then the probability of at least 2 successes is trivially equal to 0.

38 Sample space = {T, HT, HHT, HHHT, HHHHT, ...}

$$p(T) = 1/2$$

$$p(HT) = (1/2)^2 = 1/4$$

$$p(HHT) = (1/2)^3 = 1/8$$

$$p(\text{HHH} \dots \text{HT}) = (1/2)^n, n \text{ denotes the total number of coin flips}$$

Sum of probabilities of possible outcomes

$$= 1/2 + 1/4 + 1/8 + 1/16 + \dots \text{ (sum to infinity terms)}$$

$$= 1 \text{ (sum of GP to infinity, with } a = 1/2, r = 1/2)$$

Note: The above solution assumes that the coin is unbiased, and has a probability 1/2 of ending up either heads or tails when tossed.

If we generalize the problem, and assume that the probability of getting head in a coin toss is p (and $1 - p$ for getting tail),

$$p(\text{HHH} \dots \text{HT}) = p^{n-1}(1 - p), n \text{ denotes the total number of coin flips}$$

Thus, the sum of probabilities of possible outcomes

$$= (1 - p) + p(1 - p) + p^2(1 - p) + \dots \text{ (sum of infinite terms)}$$

$$= 1 \text{ (sum of GP to infinity, with } a = 1 - p, r = p)$$

40 Let N denote the number of flips until the experiment ends

$$P(N = n) = (1/2)^n$$

$$E(N) = \sum n P(N = n) \\ = \sum n(1/2)^n$$

Recall:

$$\sum x^r = 1 / 1 - x \quad (r = 0 \rightarrow \infty)$$

Differentiate both sides w.r.t. x :

$$\sum rx^{r-1} = 1 / (1 - x)^2$$

Multiply both sides by x :

$$\sum rx^r = x / (1 - x)^2$$

In this case, $x = 1/2$

$$\text{Thus, } E(N) = (1/2) / (1 - 1/2)^2 \\ = (1/2) / (1/4) \\ = 2$$

Note: This type of probability distribution is also known as the *geometric* distribution.

44 Let X denote the number of “6” that comes up in 10 rolls of a fair dice

Since all the 10 rolls of the dice are independent of one another,

$$E(X) = EX_1 + EX_2 + \dots + EX_{10}$$

Similarly,

$$\text{VAR}(X) = \text{VAR}(X_1) + \text{VAR}(X_2) + \dots + \text{VAR}(X_{10})$$

in which $X_n = 1$ if “6” comes up in the n^{th} roll of the dice, 0 otherwise

$$EX_n = 0 \times P(X_n = 0) + 1 \times P(X_n = 1) \\ = 1 \times (1/6) \\ = 1/6 \quad (\text{for all } n)$$

$$EX_n^2 = 0 \times P(X_n^2 = 0) + 1 \times P(X_n^2 = 1) \\ = 1/6 \\ = 1/6 \quad (\text{for all } n)$$

$$\text{VAR}(X_n) = EX_n^2 - (EX_n)^2 \\ = 1/6 - (1/6)^2 \\ = 5/36 \quad (\text{for all } n)$$

$$\text{VAR}(X) = \text{VAR}(X_1) + \text{VAR}(X_2) + \dots + \text{VAR}(X_{10}) \\ = 10 \times 5/36 \\ = 25/18$$

48(a) Let X denote the number of cans of soda pop filled in a day at the bottling plant
 $E(X) = 10\,000$

Using Markov's inequality,
$$\begin{aligned} P(X \geq 11\,000) &\leq E(X) / 11\,000 \\ &= 10\,000 / 11\,000 \\ &= 10/11 \end{aligned}$$

Therefore, upper bound for $P(X \geq 11\,000) = 10/11$