

### Section 3.1

8) a) Let  $p =$  I play hockey,  $q =$  I am sore the next day,  $r =$  I use the whirlpool.

We have:

1.  $p \rightarrow q$
2.  $q \rightarrow r$
3.  $\sim r$

We can derive "I did not play hockey" in the following way:

1.  $\sim q$  Modus Tollens on given statements 2 and 3.
2.  $\sim p$  Modus Tollens on given statement 1 and on derived statement 1.

8) c) Let  $D =$  set of dragonflies,  $I =$  set of insects,  $S =$  set of spiders,  $SL =$  set of creatures with six legs,  $E(x,y) = x$  eats  $y$ . Let us also take as given that there is at least one dragonfly and at least one spider. We can conclude several things (here are two examples):

1) Dragonflies have six legs.

Proof:

1.  $\forall x(x \in D \rightarrow x \in I)$  Given, statement 2
2.  $d \in D \rightarrow d \in I$  Universal instantiation (1).
3.  $\forall x(x \in I \rightarrow x \in SL)$  Given, statement 1
4.  $d \in I \rightarrow d \in SL$  Universal instantiation (3)
5.  $d \in D \rightarrow d \in SL$  Hypothetical syllogism (2,4)
6.  $\forall x(x \in D \rightarrow x \in SL)$  Universal generalization (5).

2) Some creature that does not have six legs eats some creature that has six legs.

1.  $\forall x(x \in S \rightarrow \sim x \in SL)$  Given, statement 3
2.  $\forall x \forall y(x \in S \wedge y \in D \rightarrow E(x,y))$  Given, statement 4
3.  $\forall x(x \in D \rightarrow x \in SL)$  See previous example
4.  $\exists x(x \in S)$  Given
5.  $\exists x(x \in D)$  Given
6.  $s \in S$  Existential instantiation (4)
7.  $d \in D$  Existential instantiation (5)
8.  $s \in S \rightarrow \sim s \in SL$  Universal instantiation (1)
9.  $d \in D \rightarrow d \in SL$  Universal instantiation (3)

Note: 6 and 7 create particular  $s$  and  $d$ , while 8 and 9 can use any  $s$  and  $d$ .

10.  $s \in S \wedge d \in D \rightarrow E(s,d)$  Universal instantiation (2)
11.  $\sim s \in SL$  Modus ponens (6,8)
12.  $d \in SL$  Modus ponens (7,9)
13.  $s \in S \wedge d \in D$  Conjunction (6,7)
14.  $E(s,d)$  Modus ponens (10,13)
15.  $\sim s \in SL \wedge d \in SL$  Conjunction (11,12)
16.  $\sim s \in SL \wedge d \in SL \wedge E(s,d)$  Conjunction (14, 15)

This is the desired result.

8) e) Let us conclude that Tofu does not taste good. Let  $H(x) = x$  is healthy,  $E(x) =$  you eat  $x$ ,  $G(x) = x$  tastes good.

1.  $\forall x(H(x) \rightarrow \neg G(x))$       Given
2.  $H(\text{Tofu}) \rightarrow \neg G(\text{Tofu})$       Universal instantiation (1)
3.  $H(\text{Tofu})$       Given.
4.  $\neg G(\text{Tofu})$       Modus ponens (2,3)

16. a) Let  $n$  be the number. Since  $n$  is even, there is an  $m$  such that  $n = 2m$ . Then  $n^2 = 4m^2 = 2(2m^2)$ . Then  $n^2$  is divisible by 2.

b) Here we try to prove that if a square of a number is odd, then the number is odd. Then  $n^2$  is odd. Let  $n = pq\dots r$ , where  $p, q, \dots, r$  are prime. Then  $n^2$  is  $ppqq\dots rr$ , and is not divisible by 2. Then  $p, q, \dots, r$  are not 2, which implies that  $n$  is not even, hence is odd.

c) Let us assume that the square of an even number is odd. Again, take  $n = 2m$ . Then  $n^2 = 4m^2 = 2(2m^2)$ . But that is supposed to be odd, a contradiction. Then, it is not the case that the square of an even number is odd. Hence, the square of an even number is even.

30) Let  $x$  be some integer. We have 4 cases:

- a)  $x \equiv 1 \pmod{5}$ . Then  $x^2 \equiv 1^2 \equiv 1 \pmod{5}$ .
- b)  $x \equiv 2 \pmod{5}$ . Then  $x^2 \equiv 2^2 \equiv 4 \pmod{5}$ .
- c)  $x \equiv 3 \pmod{5}$ . Then  $x^2 \equiv 3^2 \equiv 4 \pmod{5}$ .
- d)  $x \equiv 4 \pmod{5}$ . Then  $x^2 \equiv 4^2 \equiv 1 \pmod{5}$ .

Since all the cases give the desired result, the proof is complete.

32) Case 1 ( $n$  is even). Then  $\lfloor n/2 \rfloor = n/2$ ,  $\lceil n/2 \rceil = n/2$ , and  $\lfloor n/2 \rfloor \lceil n/2 \rceil = (n/2) * (n/2) = (n^2/4)$ . But since  $n = 2m$  for some  $m$ ,  $n^2 = 4m^2$ , and so  $n^2/4$  is an integer. Thus,  $\lfloor n^2/4 \rfloor = n^2/4$ . Thus,  $\lfloor n/2 \rfloor \lceil n/2 \rceil = \lfloor n^2/4 \rfloor$  when  $n$  is even.

Case 2 ( $n$  is odd). Then  $\lfloor n/2 \rfloor = n/2 - 1/2$ ,  $\lceil n/2 \rceil = n/2 + 1/2$ , and  $(n/2 - 1/2)(n/2 + 1/2) = n^2/4 - 1/4$ . Also,  $n = 2m + 1$  for some  $m$ , since  $n$  is odd. Then  $n^2/4 = (2m+1)(2m+1)/4 = m^2 + m + 1/4$ . Also,  $\lfloor m^2 + m + 1/4 \rfloor = m^2 + m = ((n-1)/2)^2 + (n-1)/2 = (n^2 - 2n - 1)/4 + n/2 - 1/2 = n^2/4 - n/2 - 1/4 + n/2 - 1/4 = n^2/4 - 1/4$ . Therefore, when  $n$  is odd, we have,  $\lfloor n/2 \rfloor \lceil n/2 \rceil = \lfloor n^2/4 \rfloor$ .

Since we have covered all possible cases, and all give the desired result, the proof is complete.

### Section 3.2

4. By induction, we prove the proposition  $P(n)$ , which is the statement that  $2 - 2(7) + 2(7^2) - \dots + 2(-7)^n = (1 - (-7)^{n+1})/4$ .

The base case is  $P(0)$ , in which case the left hand side is the single term 2 and the right hand side is  $(1 - (-7)^1)/4 = 8/4 = 2$ . Hence  $P(0)$  is true.

For the inductive step, we assume that  $P(n)$  is true and use this to prove that  $P(n+1)$  is true. In this case the left hand side is  $2 - 2(7) + 2(7^2) - \dots + 2(-7)^{n+1}$ , and using  $P(n)$  to rewrite all but the last term, this is  $(1 - (-7)^{n+1})/4 + 2(-7)^{n+1}$ . Rewriting the factor 2 as  $8/4$ , which is the same as  $(1 - (-7))/4$ , this last expression is

$$(1 - (-7)^{n+1} + (1 - (-7))(-7)^{n+1})/4 = (1 - (-7)^{n+1} + (-7)^{n+1} - (-7)^{n+2})/4.$$

Since the two terms of  $(-7)^{n+1}$  cancel, this is exactly the form needed for  $P(n+1)$ , so  $P(n+1)$  is true. By induction,  $P(n)$  is true for all nonnegative  $n$ .

28. We prove that  $n^2 \leq n!$  for  $n=0$ ,  $n=1$ , and  $n \geq 4$ , but not for  $n=2$  or  $n=3$ . For  $n=0$  and 1, this is true by inspection:  $0 \leq 1$  and  $1 \leq 1$ , respectively. For  $n=2$  we have 4 bigger than  $2!=2$  and for  $n=3$  we have 9 bigger than  $3!=6$ . For  $n \geq 4$ , we use induction to prove the statement  $P(n)$ :  $n^2 \leq n!$ . The base case is  $n=4$ , in which case we have  $16 \leq 24 = 4!$ . For the inductive step, we assume  $P(n)$ , then use this to prove  $P(n+1)$ . Expanding  $(n+1)^2$  and using  $P(n)$ , we have  $(n+1)^2 = n^2 + 2n + 1 \leq n! + 2n + 1 \leq n! + 3n! = 4n!$ , where we used  $n \geq 1$  to get  $2n+1 \leq 3n!$ . But since  $n \geq 4$ ,  $4n! \leq (n+1)!$ , so  $P(n+1)$  is valid. By induction,  $P(n)$  is true for all  $n \geq 4$ .

32. We prove that the amounts that can be formed using dimes and quarters are 0, 10, and all integers of the form  $20+5n$ , where  $n$  is a nonnegative integer. The amount 0 can be formed with no coins, and 10 can be formed with one dime. We prove the remaining amounts using induction. Let  $P(n)$  be the statement that  $20+5n$  can be formed using dimes and quarters. The base case is  $P(0)$ , which is verified by the fact that 20 can be formed with two dimes. For the inductive step, we assume that  $P(n)$  is true, then use this to prove that  $P(n+1)$  is true. Note that  $20+5(n+1) = (20+5n) + 5$ , and that  $20+5n$  can be formed from dimes and quarters since  $P(n)$  is true. If a quarter was used to form  $20+5n$ , then replace it with 3 dimes to get  $25+5n$ . Otherwise no quarters were used, and at least two dimes must have been used since  $20+5n$  is at least 20, so replace two dimes with one quarter to get  $25+5n$ . Hence  $P(n+1)$  is true, so  $P(n)$  is true for all nonnegative  $n$ .

40. By induction we prove that if  $a \equiv b \pmod{m}$ , then  $a^k \equiv b^k \pmod{m}$  for each nonnegative integer  $m$ . Let  $P(k)$  be the statement  $a^k \equiv b^k \pmod{m}$ . The base case is  $P(0)$ , which is true since  $a^0 = b^0 = 1$ . For the inductive step we assume  $P(k)$  and use this to prove  $P(k+1)$ . Now  $a^{k+1} = a^k a \equiv b^k a \pmod{m}$ , using  $P(k)$  and theorem 7 of section 2.3. Using the hypothesis  $a \equiv b \pmod{m}$  and another application of theorem 7 of section 2.3, we have  $b^k a \equiv b^k b \pmod{m}$ , and  $b^k b = b^{k+1}$ . Hence  $a^{k+1} \equiv b^{k+1} \pmod{m}$ , so  $P(k+1)$  is true. By induction  $P(k)$  is true for all nonnegative  $k$ .

### Section 3.3

4.  $f(0), f(1), f(2), f(3), f(4), f(5)$   
 a) 1, 1, 0, -1, -1, 0  
 b) 1, 1, 1, 1, 1, 1  
 c) 1, 1, 2, 5, 33, 1214

d) 1, 1, 1, 1, 1, 1

18. Define  $\max\{a_1\}$  to be  $a_1$ . Recursively, for  $n$  bigger than 0, define  $\max\{a_1, \dots, a_{n+1}\}$  to be  $a_{n+1}$  if  $a_{n+1} > \max\{a_1, \dots, a_n\}$  and  $\max\{a_1, \dots, a_n\}$  otherwise.

Likewise, define  $\min\{a_1\}$  to be  $a_1$  and recursively, for  $n$  bigger than 0, define  $\min\{a_1, \dots, a_{n+1}\}$  to be  $a_{n+1}$  if  $a_{n+1} < \min\{a_1, \dots, a_n\}$  and  $\min\{a_1, \dots, a_n\}$  otherwise.

22. a)  $a_1 = 1$ ,  $a_{n+1} = a_n + 2$  if  $n > 0$ .

b)  $a_1 = 3$ ,  $a_{n+1} = 3a_n$  if  $n > 0$ .

c) (In this part,  $x$  represents the symbolic variable used to define a polynomial.) For all integers  $n$ ,  $n$  is an element of  $S$ . If  $p$  is an element of  $S$  and  $n$  is an integer, then  $x^p + n$  is an element of  $S$ .

40. Using the recursive definition, we have  $A(3,4) = A(2, A(3,3))$ . Now,  $A(3,3) = A(2, A(3,2))$ , and  $A(3,2) = A(2, A(3,1)) = A(2,2) = 4$  (by part 3 of the def and problem 37). So,  $A(3,3) = A(2, 4) = A(1, A(2,3))$ , and  $A(2,3) = A(1, A(2,2)) = A(1,4) = 16$  (by problem 38). So,  $A(3,3) = A(1, 16) = 2^{16} = 65536$  by problem 38.

Hence  $A(3,4) = A(2, 65536) = A(1, A(2, 65535)) = 2^{A(2, 65535)}$ .  $A(2, 65535) = A(1, A(2, 65534)) = 2^{A(2, 65534)}$ . Continuing until  $A(2,2)=4$ , we see that  $A(3,4)$  is a tower of powers of 2 with 65536 2's in the tower!

To state the answer another way, let  $a_1=2$  and define  $a_{n+1}$  recursively by  $a_{n+1}$  is 2 raised to the power  $a_n$ . Then  $A(3,4) = a_{65536}$ .

### Section 3.4

6. We assume that  $x \bmod m$  computes the remainder when  $x$  is divided by  $m$ . Note that at each step we take the result mod  $m$  so that we never have to deal with numbers larger than the max of  $m*x$  and  $x*x$ .

**procedure** modPower( $x$ ,  $n$ ,  $m$ : positive integers)

if  $n=1$  then  $y:=x \bmod m$  // In the base case, get the remainder.

else

begin

$y:=\text{modPower}(x, n-1, m)$  // Make the recursive call to get  $y = x^{n-1} \bmod m$

$y:=y*x$  // Multiply for the extra power of  $x$

$y:=y \bmod m$  // Take the remainder.

end

return  $y$  // Return the result

end

22.

**procedure** partition(m: positive integer)

if m=1 then y:=1

else y:=partitionAux(m, m-1)

return y

end

**procedure** partitionAux(m, n: positive integers)

if m=1 then y:=1

else if n=1 then y:=1

else if m<n then y:=partitionAux(m, m)

else if m=n then y:=1+partitionAux(m, m-1)

else if m>n then y:=partitionAux(m, n-1) + partitionAux(m-n, n)

return y

end