SOLUTIONS

Section 7.5

12

It does. In fact it even has an Euler circuit. This is noted by the simple fact that every node in the graph has an even degree, a necessary and sufficient requirement for having a circuit. An example of such a path is e-d-c-e-b-c-d-b-a-e-a-e. This path clearly covers every edge. No edge is covered more than once.

20

Convert this image into a graph: Label every intersection by a vertex. Label every corner by a vertex. Also label the two endpoints with a vertex each. Then every intersecting node has a degree of 4. Every corner vertex has a degree of 2 and there are exactly two nodes that have a degree of 1 (an odd degree). All the edges of the graph are exactly all the lines of the picture and so the problem of whether or not it is possible to draw the picture without retracing it in a continuous motion is exactly the problem of determining whether or not an Euler path exists in the graph.

There are nodes with odd degree and so by Theorem 1 in 7.5 we conclude that no circuit exists. What about a path? We need not look further than Theorem 2 in the same section. Because we have exactly two nodes with an odd degree, we conclude that a path exists and thus, the figure can be drawn in the manner specified.

36a K_n

In a complete graph, every vertex has the same degree: n-1. Since the requirement for the existence of a Euler circuit is that each degree is even, we conclude that the only requirement for a complete graph to contain an Euler circuit is that n must be odd.

36d Q_n

Consider a node in an n-cube. It is represented by a n-sized bit string. It is connected to all those nodes whose bit-string representation differs from it in exactly one place. Therefore, there is a one-to one correspondence between a node's neighbors and the indices of bits in its string representations. Since the latter has the cardinality of n, we conclude that every vertex in the graph has exactly n neighbors. Theorem 1 of 7.5 forces requires n to be even for it to contain an Euler circuit.

38

(*Circuits*) In a bipartite graph, there are m nodes that have degree n and the other n nodes that have degree m. So in order for the graph to contain an Euler circuit, we must have both m and n to be even (as theorem 1 requires).

(*Paths*) The above conditions also holds for the paths, except there is a case when exactly two nodes are allowed an odd degree (by theorem 2). This happens exactly when n (without loss of generality) is odd (so that m nodes have an odd degree n) and m is exactly 2 (so that exactly 2 nodes have the odd degree n). Every other node (any of the n nodes) has an even degree of m = 2. Thus the requirement for having an

Euler path is that either both *m* and *n* are even OR if one of them is odd, the other better be exactly 2.

Section 7.6

6a
path between a-d

A shortest path between a and d is the path from a to c to d of weight 6. Every path to d must either go through b, which leads to a weight of at least 9, or through c, which leads to a weight of at least 10.

6b
path between a-f

A shortest path from a to f is the path from a to c to d to f of weight 11.

18

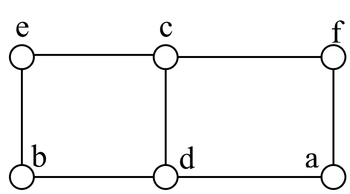
The shortest path may not be unique. Consider the subgraph of the figure in exercise 4 on pg.499 that involves nodes a, c, d and f. There are only two paths from a to f. Both are of weight 6 and so both are minimal. All of the weights are distinct: 1,5,2,4. So even though weights are unique, more than one shortest path may exist.

Section 7.7

6

It clearly has no K_5 or $K_{3,3}$ contained within it, and so is not nonplanar (by Theorem 2)

This graph is planar can be redrawn as



16

Consider graph G as a collection of individual connected components: $G_i=(V_i, E_i)$. Since the graph itself is planar, each of the components must be as well, and so by Euler formula we have that

$$r_i = e_i - v_i + 2.$$

However each of the r terms includes an unbounded region that all share. Thus for every additional component we overcount that region once:

$$r = (\Sigma r_i) - k + 1.$$

Since all E and V are disjoint in the connected components we can simply add their cardinalities. Thus

$$r = e - v + 2k - k + 1 = e - v + k + 1.$$

Note, for $k = 1$ we get the Euler formula, as we should.

Section 7.8

12

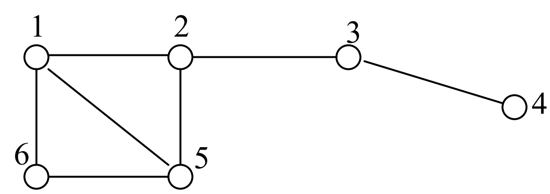
We are clearly dealing with a planar graph, so the chromatic number is at most 4. Consider the states California, Oregon, Idaho, Utah, Arizona, and Nevada. The graph associated with these states is the wheel with 5 spokes and Nevada in the center. As shown in example 4 in section 7.8, the cycle of length 5 requires 3 colors. Since this wheel consists of the cycle of length 5 plus a central vertex adjacent to each of the 5 vertices in the cycle, the central vertex must use a color different from each of the vertices in the spokes, so 4 colors are required.

14

Suppose it can. Suppose (without loss of generality) the colors are white and black and pick any node on the circuit, coloring it white. Then if we call that node to be the 1st node on the circuit, then every even node is black and every odd node is white, since colors alternate as we progress through the circuit. But then the last node, which is odd is white and the first node which neighbors it (as the path is closed) is also white. So the coloring assignment we picked is not a coloring. This is a contradiction to the assumption that a 2-coloring exists.

16

Draw a graph with 6 vertices representing stations and connect them with edges if they are closer than 150 miles. Then the number of channels required is the chromatic number of the graph.



The chromatic number of this graph is 3: it cannot be 2 since K3 is a subgraph. 3 is possible—color 2,6,4 into red, 3 and 5 into white and 1 into blue.

20

c) C_n

if n is even then clearly the edge chromatic number is 2 as we must alternate the colors as we go from edge to edge in the cycle. If n is odd, however, the number is 3 as the last edge in the cycle cannot be of the same color as the first nor the previous one (which are the same color)

d) W_n

Since there are n edges coming out of the central vertex, at least n colors must be used. To obtain a coloring with n colors, label the edges along the rim with colors 1 through n in order. Then for the spoke into the vertex with colors 1 and 2, use color n. For the spoke into the vertex with colors 2 and 3, use color 1, and in general for the spoke into the vertex with colors j and j+1, use color j-1, using n-2 for the spoke into the vertex with colors n and 1. Then exactly n colors are used and no two edges that share a vertex have the same color.