Discrete Math in CS	**Problem Set 7**
CS 280 Fall 2000	**Due Monday, November 6, 2000**

Reading: Rosen, Sections 7.1-7.4, 7.7-7.8, 8.1, 8.3, 8.5

To use the book's terminology, all graphs considered below are *simple*; they do not have loops or parallel edges. Also, as in class, "graph" by default means "undirected graph"; if a graph is meant to be directed, this will be explicitly noted.

As in previous problem sets, some of the questions present "claims," and ask you to decide whether they are true or false. For such a question, you should do one of the following two things: (i) give a proof of the claim; or (ii) give a proof that the claim is false, by showing that its negation is true. (Using a counter-example or other means.)

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(1) A number of recent stories in the press about the structure of the Internet and the Web have focused on some version of the following question: How far apart are typical nodes in these networks? If you read these stories carefully, you find that many of them are confused about the difference between the *diameter* of a network and the *average distance* in a network — they often jump back and forth between these concepts as though they're the same thing.

As in class, we say that the *distance* between two nodes u and v in a graph $G = (V, E)$ is the minimum number of edges in a path joining them; we'll denote this by $dist(u, v)$. We say that the *diameter* of G is the maximum distance between any pair of nodes; and we'll denote this quantity by $diam(G)$.

Let's define a related quantity, which we'll call the *average pairwise distance* in G (denoted $apd(G)$). We define $apd(G)$ to be the average, over all $\binom{n}{2}$ sets of two distinct nodes u and v, of the distance between u and v. That is,

$$apd(G) = \left[\sum_{\{u,v\} \subseteq V} dist(u, v) \right] / \binom{n}{2}.$$

Here's a simple example to convince yourself that there are graphs G for which $diam(G) \neq apd(G)$. Let G be a graph with three nodes u, v, w; and with the two edges $\{u, v\}$ and $\{v, w\}$. Then

$$diam(G) = dist(u, w) = 2,$$

while

$$apd(G) = [dist(u, v) + dist(u, w) + dist(v, w)]/3 = 4/3.$$

Of course, these two numbers aren't all *that* far apart in the case of this 3-node graph, and so it's natural to ask whether there's always a close relation between them. Here's a claim that tries to make this precise.

Claim: There exists a positive natural number c so that for all graphs G, it is the case that

$$\frac{diam(G)}{apd(G)} \leq c.$$

Decide whether you think the claim is true or false, and give a proof of either the claim or its negation.

(2) Some friends of yours work on wireless networks, and they're currently studying the properties of a network of n mobile devices. As the devices move around (really, as their human owners move around), they define a graph at any point in time as follows: there is a node representing each of the n devices, and there is an edge between device i and device j if the physical locations of i and j are no more than 500 meters apart. (If so, we say that i and j are "in range" of each other.)

They'd like it to be the case that the network of devices is connected at all times, and so they've constrained the motion of the devices to satisfy the following property: at all times, each device i is within 500 meters of at least $n/2$ of the other devices. (We'll assume n is an even number.) What they'd like to know is: Does this property by itself guarantee that the network will remain connected?

Here's a concrete way to formulate the question as a claim about graphs:

Claim: Let G be a graph on n nodes, where n is an even number. If every node of G has degree at least $n/2$, then G is connected.

Decide whether you think the claim is true or false, and give a proof of either the claim or its negation.

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(3) As in class, we say that a *k-coloring* of a graph $G = (V, E)$ is an assignment of one of k labels to each node, in such a way that for every edge $\{u, v\} \in E$, the nodes u and v receive different labels. We say that G is *k-colorable* if there exists a k-coloring of G.

Many heuristics for constructing colorings try to build up a coloring for a graph G from colorings of its subgraphs. Thus, we want to understand the extent to which this is possible. One natural question is contained in the following claim.

Claim: Let $G = (V, E)$ be any graph with n nodes, and let k be any natural number. If every subgraph of G consisting of $n - 1$ nodes is k-colorable, then G is k-colorable.

Decide whether you think the claim is true or false, and give a proof of either the claim or its negation.

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(4) *Fault-tolerance* is another large area of research in networks — to what extent can a given network withstand the failure of nodes and edges? For example, will the phone system keep working even when a construction crew accidentally cuts through a major fiber-optic cable? Will the Internet keep working even when a power outage brings down half the computers in the Washington D.C. area?

When we formulate such issues in terms of graphs, we get questions about the behavior of graphs when various portions of them are deleted. Here's a basic question in this spirit.

Let $G = (V, E)$ be a connected graph. We say that G is *bridge-free* if for every edge e, the subgraph $(V, E - \{e\})$ is connected. (In other words, if G remains connected after the deletion of any one of its edges.) We say that G is *robust* if G has spanning trees T_1 and T_2 with the property that T_1 and T_2 do not have any edges in common.

Both of these definitions relate to the "resilience" of the graph G. But are they really the same concept? Consider the following claim.

> *Claim: Let $G = (V, E)$ be any connected graph. Then G is bridge-free if and only if it is robust. (In other words: if G is bridge-free, then it is robust;* and *if G is robust then it is bridge-free.)*

Decide whether you think the claim is true or false, and give a proof of either the claim or its negation.

(5) We have a connected graph $G = (V, E)$, and a specific vertex $u \in V$. Suppose we compute a depth-first search tree rooted at u, and obtain the spanning tree T. Suppose we then compute a breadth-first search tree rooted at u, and obtain the same spanning tree T. Prove that $G = T$. (In other words, if T is both a depth-first search tree and a breadth-first search tree rooted at u, then G cannot contain any edges that do not belong to T.)