Discrete Math in CS	**Problem Set 3**
CS 280 Fall 2000	**Due September 22, 2000**

Reading: Rosen, Sections 2.3 – 2.5.

As in previous problem sets, some of the questions present "claims," and ask you to decide whether they are true or false. For such a question, you should do one of the following two things: (i) give a proof of the claim; or (ii) give a proof that the claim is false, by showing that its negation is true. (Using a counter-example or other means.)

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(1) *(Adapted from an example of L. Lovász.)* Consider the following claim, with a purported proof.

> *Claim: For all $n \geq 1$, the number $n(n+3)$ is odd.*
>
> *"Proof": Let $P(n)$ denote the statement "$n(n+3)$ is odd." We prove $P(n)$ for all natural numbers $n \geq 1$. Suppose that $P(m)$ is true for all values of m from 1 through n; we must show that this implies $P(n+1)$.*
>
> *To establish $P(n+1)$, we must show that $(n+1)(n+1+3) = (n+1)(n+4)$ is odd. We can write this as*
>
> $$(n+1)(n+4) = n(n+3) + 2n + 4.$$
>
> *Now, by induction hypothesis, $n(n+3)$ is odd. (This is just the statement $P(n)$.) We then add the numbers $2n$ (which is even) and 4 (which is even), and so the entire right-hand side of the equation is odd.*
>
> *Thus the left-hand side of the equation is odd. This shows that $P(n+1)$ holds, and completes the proof.*

But this claimed proof must somehow be wrong, since $n(n+3)$ is not odd for all n. For example, if $n = 5$, then $n(n+3) = 40$, which is even.

Give an explanation why the claimed proof is incorrect.

(2) For each natural number $n \geq 1$, define

$$f(n) = \sum_{i=1}^{n} \frac{i}{(i+1)!} = \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!}.$$

Prove that for all $n \geq 1$, $f(n) = 1 - \dfrac{1}{(n+1)!}$.

(3) Recall the definition of a *locally fair ranking* from class. We have a tournament with n teams, and each pair of teams plays exactly once, resulting in a win for one of the teams and a loss for the other. (There are no ties.) A *ranking* is any ordering of the teams $T[1], T[2], \ldots, T[n]$. A ranking is *locally fair* if for each i from 1 through $n-1$, the team $T[i]$ beat the team $T[i+1]$. (So $T[1]$ beat $T[2]$, $T[2]$ beat $T[3]$, and so on.) Note that a locally fair ranking involves no condition on what happened between $T[i]$ and $T[j]$ when i and j are not numbers differing by exactly one.

Suppose a tournament has the property that there exists a single team X that won more games than any other team. Prove that there exists a locally fair ranking in which X appears first in the order. (That is, there is a locally fair ranking $T[1], T[2], \ldots, T[n]$ in which $T[1] = X$.)

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(4) Consider the following claim.

> *Claim: For every k, there exist k consecutive natural numbers (i.e. k numbers of the form $n+1, n+2, n+3, \ldots, n+k$ for some n) so that none of them is prime.*

Decide whether you think the claim is true or false, and give a proof of either the claim or its negation.

(5) We define a *triple prime sequence* to be a sequence of three natural numbers (a, b, c) so that $b = a + 2$, $c = b + 2$, and each of a, b, and c is a prime number. For example, $(3, 5, 7)$ is a triple prime sequence; $(1, 3, 5)$ is not a triple prime sequence since 1 is not a prime number.

Now consider the following claim.

> *Claim:* $(3, 5, 7)$ *is the only triple prime sequence.*

Decide whether you think the claim is true or false, and give a proof of either the claim or its negation.

Given natural numbers a and b, with $a \geq 0$ and $b \geq 1$, we define the set $S(a, b)$ to consist of all numbers of the form $a + bn$, as n ranges over all natural numbers. That is,

$$S(a, b) = a, a + b, a + 2b, a + 3b, \ldots$$

To take an example, $S(1, 2)$ is all the positive odd numbers: $S(1, 2) = 1, 3, 5, 7, \ldots$

We say that a set of natural number is an *infinite arithmetic progression* if it is equal to $S(a, b)$ for some choice of natural numbers a and b.

(6) Consider the following claim.

> *Claim: There is an infinite arithmetic progression, all of whose elements are prime numbers.*

Decide whether you think the claim is true or false, and give a proof of either the claim or its negation.