Discrete Math in CS	**Problem Set 2**
CS 280 Fall 2000	**Due September 15, 2000**

Reading: Rosen, Section 3.2 and pages 202-205 in Section 3.3.

Note: Questions 3-6 present claims, all of them true, and ask you to provide proofs. Each of these can be proved by induction. For each induction proof, you should clearly state: (i) the variable over which you're applying induction, and the statement that you're proving by induction, in terms of this variable; (ii) the proof of the base case; (iii) the induction hypothesis; and (iv) the proof of the inductive step.

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(1) For each of the following English sentences, write an English sentence that is equivalent to its logical negation. While it will help to convert the sentence into quantifiers and Boolean connectives in order to take the negation, you do not need to hand this in with your answer. Your final answer should consist just of an English sentence.

Also, your answer should not simply begin, "It is not the case that ..." (or some equivalent phrasing) and then just repeat the original sentence.

Note: Inevitably, sentences written in English can be ambiguous. For sentences you think are ambiguous, explain what you see as its two (or more) possible meanings, and explain which one you are using in forming the negation. Also, note that the question does not require you to actually decide whether you think the sentence is true or false.

Example: Suppose the sentence is, "In every month this year, the market value of Akamai Technologies has exceeded the market value of Sears Roebuck." Then a correct answer would be, "In some month this year, the market value of Sears Roebuck was at least as large as the market value of Akamai Technologies." Personally, I don't see any ambiguity — i.e. alternate interpretations — for this sentence.

(You might arrive at such an answer by first converting the sentence to pseudo-mathematical notation like, "$\forall$ months in this year: $value(Akamai) > value(Sears)$." Then, negating the quantified sentence, you'd get "$\exists$ month in this year: $value(Sears) \geq value(Akamai)$." Finally, converting this to an English sentence, you'd get a correct answer. Again, you should only write the final sentence you obtain.)

The sentences:

1. Every natural number is a multiple of 2 or 3.

2. Every point in England is within 75 miles of the ocean.

3. Every point in England is within 15 miles of a beach or a museum.

4. For every query it receives, Google responds within 2 seconds or reports an error message.

5. In each of the last three years, Microsoft has sold at least 75% of all the personal computer software in the world.

6. For every piece of environmental legislation this year, at least ten U.S. Senators voted against it.

7. Everyone who lives in Risley knows someone who lives in Collegetown.

(2) For each of the following Boolean expressions Φ, write a Boolean expression Ψ with the following properties: (i) Ψ is logically equivalent to Φ; and (ii) each variable appears at most once in Ψ. Use a truth table to demonstrate that they are logically equivalent.

Example: Suppose the expression Φ is $\neg p \wedge (p \vee q)$. Then an equivalent expression Ψ would be $\neg p \wedge q$. Note that each of the symbols p and q appears at most once in Ψ, and the following truth table shows that Ψ is logically equivalent to Φ:

p	q	$\neg p \wedge (p \vee q)$	$\neg p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	F	F

(a) $p \longrightarrow [p \wedge (q \wedge r)]$
(b) $[(p \wedge q) \longrightarrow (p \vee r)] \longrightarrow (q \longrightarrow r)$.

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(3) For each natural number $n \geq 1$, define

$$f(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} + \cdots + \frac{1}{n(n+1)}.$$

Prove that for all $n \geq 1$, $f(n) = \dfrac{n}{n+1}$.

(4) (a) Consider a sport in which two teams play each other, and can accumulate points only in incremements of 3 and 7. (You're welcome to draw analogies to (American) football if you want, though this is not necessary.) Prove that in this sport, it is possible for a team to end the game with n points, for any natural number $n > 11$.

Example: Two teams A and B play each other. First A scores 7 points three times in a row, then B scores 7 points two in a row, and then B scores 3 points. So A ends with 21 points and B ends with 17 points; this implies that 17 and 21 are possible scores.

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(5) First, a definition and a fact. Many of you have the seen the function $n!$ (n factorial) from calculus or intro computer science; it is the product of all the natural numbers from 1 through n. Alternately, we can view it as having the following recursive definition:

(i) $1! = 1$.

(ii) For $n \geq 1$, $(n+1)! = (n+1) \cdot n!$.

It's not immediately clear how to understand the magnitude of $n!$. It's clearly less than n^n (since multiplying n copies of n together will give you a bigger number than multiplying all the numbers from 1 to n together). But how would it compare with $\left(\frac{n}{4}\right)^n$, for example? Would you get a bigger number by multiplying together n copes of $n/4$, or all numbers from 1 to n?

To resolve questions like this, we recall the following fact from calculus, which you can use without proving.

Fact A: The number $e \approx 2.71828\cdots$ *satisfies the inequality*

$$\left(1 + \frac{1}{n}\right)^n < e$$

for every natural number n*; and it is the smallest real number to have this property.*

Prove that for all natural numbers $n \geq 1$,

$$n! > \left(\frac{n}{e}\right)^n.$$

You can use Fact A in your proof.

Note: In fact, e is the smallest real number x with the property that $n! > (n/x)^n$ holds for all n. You could think about how one might go about proving this; but you don't have to hand anything in along these lines.

(6) Consider the following two-player game. n coins are placed on a table, and the two players take turns removing coins from the table. In a single turn, a player can choose to remove either 1 or 2 coins. (View all coins as being indistinguishable in this game.) The player that removes the final coins (i.e. the player whose turn ends with the table first being empty) wins. Notice that unlike tic-tac-toe or chess, this game cannot end in a draw — one player or the other must win.

If we think about directly applying the game-tree approach we used in lecture for a large value of n (like 100), we'd end up with a structure of enormous complexity. And yet, it turns out we can understand this game completely by a different approach: we find a simple-to-state "pattern" that explains who wins, and prove it by induction.

Prove that if n is not divisible by 3, then the player who moves first can force a win; and if n is divisible by 3, then the player who moves second can force a win.

Note: So we now know exactly the situations under which each player can force a win. Just by way of reassurance, an industrial-strength computer game-playing program wouldn't build the entire search tree from lecture either. We won't go into the details of this here, but it uses the technique of *memoization* (also called "transposition tables" in computer chess), which is covered in CS 312 and 482, and is sometimes a topic in CS 211.

Optional 7th Question. You don't need to hand anything in for this, but I mention it in case you find it interesting.

Consider a two-player game in which m nickels and n dimes are placed on a table, and the two players take turns removing coins from the table. Player 1 moves first. In a single turn, a player can choose to remove 1 or 2 nickels, or 1 or 2 dimes. (So there are four possible moves; a player cannot remove both a nickel and a dime in one turn.) Again, the player whose turn ends with the table first being empty wins.

For which values of (m, n) can Player 1 force a win? For which values of (m, n) can Player 2 force a win?