Discrete Math in CS	Problem Set 1
CS 280 Fall 2000	Due September 8, 2000

Reading: Rosen, Sections 1.1 – 1.3, and Section 3.1. Also "Boolean Formulas" handout. For examples of the style in which we want you to write proofs, see the proofs in the handout and in the book (especially pages 175-181).

Note: Some of the questions present "claims," and ask you to decide whether they are true or false. For such a question, you should do one of the following two things: (i) give a proof of the claim; or (ii) give a counter-example that demonstrates the claim is false, together with an explanation of your counter-example.

In addition to correctness, solutions involving proofs or explanations will be graded on style and clarity.

---

(1) Construct truth tables for the following Boolean expressions (as in Section 1.1 of Rosen's book.)

(a) $(p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee q)$.

(b) $(p \wedge q) \longrightarrow (\neg p \vee \neg q)$.

(2) Another use of Boolean formulas is for designing classification rules in the fields of artificial intelligence, data mining, and statistics. For example, a direct-mail marketing firm might have customer data on millions of individuals throughout the U.S.; when it has a new offer to promote, it has to decide which of these customers is likely to be interested. In other words, it must *classify* the entries in its customer database into "likely" or "unlikely," with the goal of answering the question: Is a given customer likely enough to buy the product that it's worth spending the cost of mailing an ad to them?

Often, such a classification rule is a predicate on individual customers, expressed as a Boolean formula.

Here's a concrete example. Suppose a marketer of lutes and other medieval musical instruments has a database of Cornell students and alumni. It's too expensive to send a copy of their catalog to everyone in the database, so an in-house consultant suggests three candidates for classification rules they can use in deciding who gets mailed a catalog.

Here are the possible rules, expressed as predicates in which x is variable denoting a person in the database.

- *Rule 1:* (x lived in Risley sometime during the years 1995-2000) AND (x's major is Music OR x's major is Theater Arts).
- *Rule 2:* (x lived in Risley sometime during the years 1997-2000) AND (x's major is Music OR x's major is Theater Arts OR x's major is Computer Science).

- *Rule 3:* (x lived in Risley at some point during his or her life) AND (x's major is Music).

(a) Give an example of a (hypothetical) person who would satisfy Rule 1 but neither of Rules 2 or 3.

(b) Give an example of a (hypothetical) person who would satisfy Rule 2 but neither of Rules 1 or 3.

(c) Give an example of a (hypothetical) person who would satisfy Rule 3 but neither of Rules 1 or 2.

Recall the definitions of positive coalitions and negative coalitions from the handout on Boolean formulas. For a Boolean formula F, we define $p(F)$ to be the smallest number of variables in any positive coalition in F. We define $n(F)$ to be the smallest number of variables in any negative coalition in F. So for example, if F is a Boolean formula consisting of a single $\wedge$-gate that takes two inputs, then $p(F) = 2$ and $n(F) = 1$. (This is a consequence of (3.1) from the handout.)

(3) Let F be the formula depicted in Figure 1; it has 40 variables labeled "1," "2," ... "40" in the figure. Determine the value of $p(F)$, and write down a positive coalition containing exactly $p(F)$ variables. You do not need to provide any explanation for your answer.

(*Note:* There are far too many distinct subsets of variables to try them all explicitly. Instead, think about how the value $p(F)$ can be built up from small portions of the full formula F.)

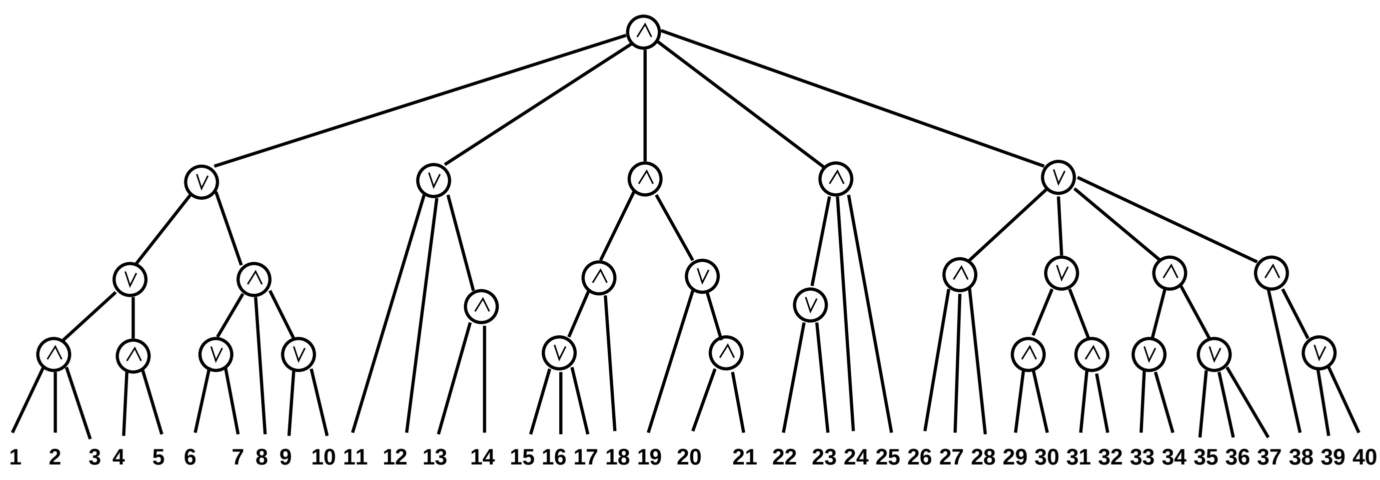

Figure 1: A formula F with 40 variables.

(4) Consider the following claim.

> *Claim: If F is a Boolean formula on at least two variables, and the top gate of F is an $\wedge$-gate, then $p(F) > 1$.*

Is this true or false? If you believe it is true, give a proof of the claim. If you believe it is false, disprove it by giving an example of a Boolean formula F on at least two variables, whose top gate is an $\wedge$-gate, and for which $p(F) \leq 1$.

In lecture, we considered the idea of an *assignment* A to the variables in a Boolean formula; such an assignment maps each variable to one of the values 0 or 1.

It's also useful to think about situations in which only some of the variables are given values, and the rest are left "unspecified." Let F be a Boolean formula over a set X of variables. We define a *partial setting* P of X to be a function that maps each variable to one of the values 0, 1, or ?. We say that a variable is *specified* by P if it is mapped to one of the values 0 or 1; we say that a variable is *unspecified* by P if it is mapped to the value ?. Thus, an *assignment* (under the definition from lecture) is simply a partial setting in which each variable is specified.

Given two partial settings P and P', we say that they are *consistent* with each other if, for every variable x that is specified by both P and P', x is mapped to the same value by P and P'. We say that a partial setting P is *decisive* if F has the same value with respect to every assignment A that is consistent with P. (In other words, a partial setting is decisive if, when the values of its specified variables are revealed, the value of F is determined regardless of how any unspecified variables are set.)

So for example, suppose F is a Boolean formula consisting of a single $\wedge$-gate that takes the three inputs x_1, x_2, and x_3. Suppose P is the partial setting that maps x_1 to 0, x_2 to 1, and x_3 to ?. Suppose P' is the partial setting that maps x_1 to ?, x_2 to 1, and x_3 to 1. Then P and P' are consistent with each other, since the only variable specified by both P and P' is x_2, and it is mapped to the same value (i.e. to 1) by both P and P'. Moreover, P is decisive, since in any assignment A consistent with P, the variable x_1 is mapped to 0, and hence the value of F is 0.

(Continued on next page.)

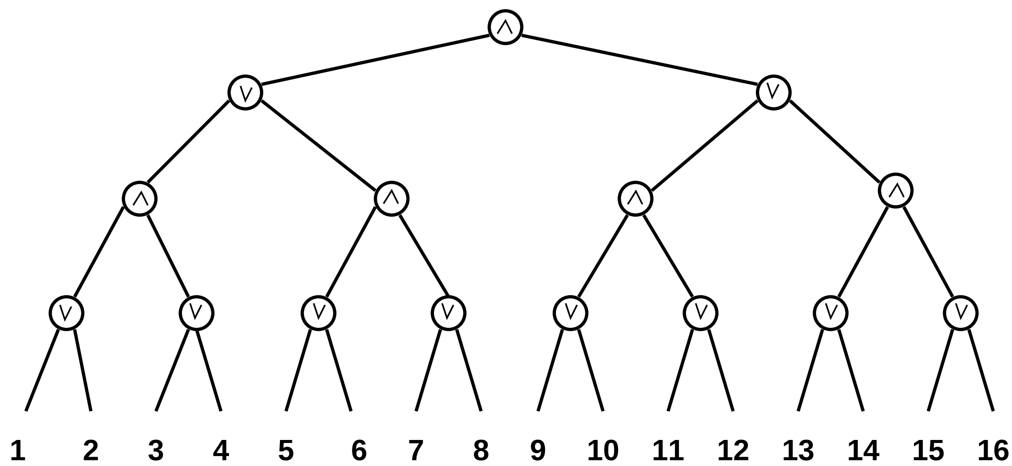

Figure 2: A formula with 16 variables.

(5) Consider the following claim.

> *Claim: Let B denote the formula in Figure 2; it has 16 variables labeled "1," "2," ... "16" in the figure. If P is a partial setting with the property that each of variables "1," "2," ... "15" is specified by P, then P is decisive.*

Is this true or false? If you believe it is true, give a proof of the claim. If you believe it is false, disprove it by giving an example of a partial setting P that is not decisive, but has the property that each of the variables "1," "2," ... "15" is specified by P.

(6) Consider the following claim.

> *Claim: Let F be a Boolean formula on a set X of at least two variables, and let $P_1, P_2, \ldots, P_k$ be k different partial settings of X. If every two of these partial settings $P_1, P_2, \ldots, P_k$ are consistent with each other, then there exists an assignment A to X that is consistent with each one of $P_1, P_2, \ldots, P_k$.*

Again, is this true or false? If you believe it is true, give a proof of the claim. If you believe it is false, disprove it by giving an example of a formula F on at least two variables, and partial settings $P_1, P_2, \ldots, P_k$ so that each two are consistent with each other, but for which there is no assignment A that is consistent with each one of them.