CS 222: Introduction to Scientific Computing
Spring 2001
Practice Prelim 2

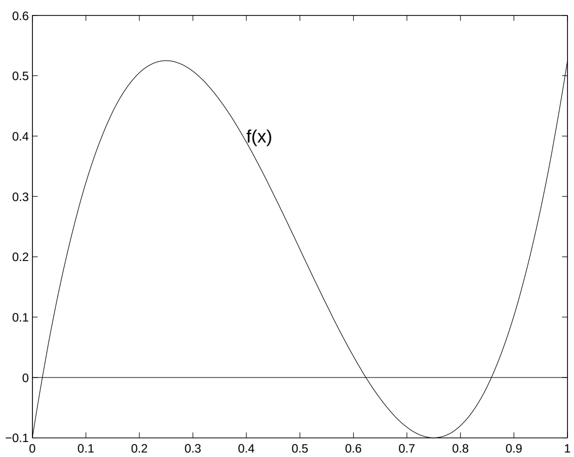
Handed out: Tues., Apr. 3.

This examination lasted 80 minutes and had 80 points total. It was closed book and closed note, but students were permitted to use a $8\frac{1}{2}'' \times 11''$ crib-sheet with notes written on both sides.

1. **[5 points]** For an $n \times n$ symmetric matrix A, what is the definition of "positive definite"?

2. **[5 points]** Which algorithm should be used to solve the linear least squares problem of minimize $\|R\mathbf{x} - \mathbf{b}\|_2$, where R is an $m \times n$ upper triangular matrix with nonzero entries on its main diagonal? How many flops are required (accurate to the leading term)?

3. **[10 points]** For the function plotted $f(x)$ below, mark the first three iterates of the bisection method (draw three ticks on the x-axis and label them x_1, x_2, x_3) over the interval $[0, 1]$ and circle the root of the function to which the bisection method eventually converges.



4. **[10 points]** Come up with an example of a 2×2 matrix A such that A is symmetric, all of its entries are positive, and yet A is not positive definite. Note: for full credit, in addition to providing A, also provide a demonstration that A is not positive definite.

5. **[10 points]** Given a symmetric positive definite $n \times n$ matrix A and a sequence of $n/2$ vectors $\mathbf{b}_1, \ldots, \mathbf{b}_{n/2}$ each lying in $\mathbf{R}^n$, consider the problem of finding $\mathbf{x}_1, \ldots, \mathbf{x}_{n/2}$ each in $\mathbf{R}^n$ that solve the linear systems $A\mathbf{x}_1 = \mathbf{b}_1, \ldots, A\mathbf{x}_{n/2} = \mathbf{b}_{n/2}$. Propose an efficient algorithm for computing $\mathbf{x}_1, \ldots, \mathbf{x}_{n/2}$, and determine the number of flops (accurate to the leading term) it requires. You may describe your algorithm at a very high level using algorithms described in lecture (such as Cholesky, forward substitution, etc.)

6. **[15 points]** The 1-dimensional case of the "Brouwer fixed point theorem" states that if f is a continuous function from $[0,1]$ to $[0,1]$, then there exists a point x^* in $[0,1]$ such that $f(x^*) = x^*$. Rewrite the problem of finding x^* as a root-finding problem (hint: consider $f(x) - x$), and then argue that the bisection method can find x^*, i.e., argue that the necessary conditions for the bisection method are applicable to this problem.

7. **[15 points]** How many square-root operations (accurate to the leading term) are necessary to reduce an $m \times n$ matrix ($m \geq n$) to upper triangular form using Givens rotations? Recall that to compute a single Givens rotation, one square-root operation is required.

8. **[15 points]** What is the solution to the linear least-squares problem $\min \|\mathbf{a}x - \mathbf{b}\|_2$, where $\mathbf{a}, \mathbf{b}$ are given n-vectors and x is an unknown scalar? Come up with a closed-form solution for x. [Hint: consider the normal equations.]