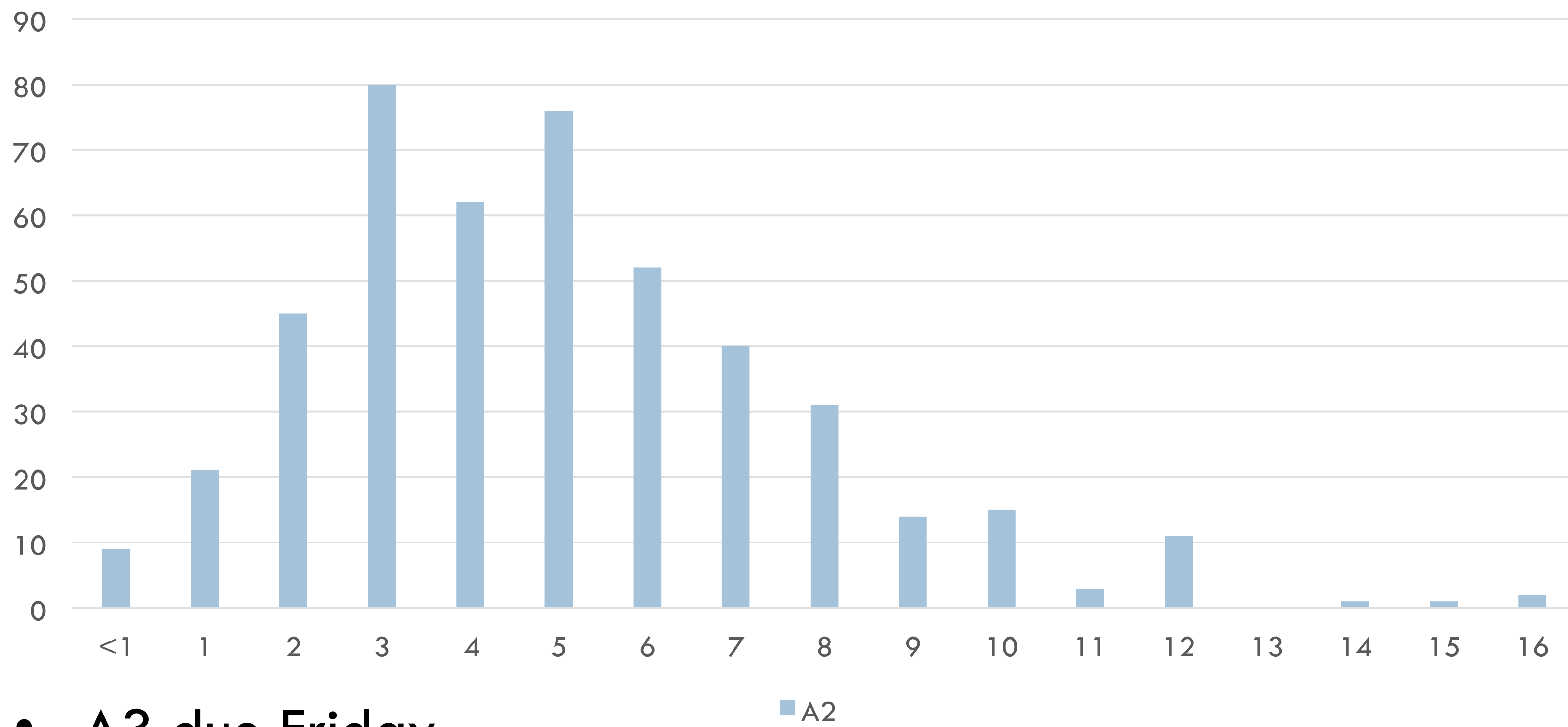
“Progress is made by lazy men looking for easier ways to do things.”

- Robert Heinlein

ASYMPTOTIC COMPLEXITY

Lecture 10
CS2110 – Fall 2017

Announcements



- A3 due Friday
- Prelim next Thursday
  - Prelim conflicts: fill out CMS by Friday
  - Review section on Sunday

What Makes a Good Algorithm?

Suppose you have two possible algorithms that do the same thing; which is *better*?

What do we mean by *better*?

- Faster?
- Less space?
- Easier to code?
- Easier to maintain?
- Required for homework?

FIRST, Aim for simplicity, ease of understanding, correctness.

SECOND, Worry about efficiency only when it is needed.

How do we measure speed of an algorithm?

Basic Step: one "constant time" operation

Constant time operation: its time doesn't depend on the size or length of anything. Always roughly the same. Time is bounded above by some number

Basic step:

- Input/output of a number
- Access value of primitive-type variable, array element, or object field
- assign to variable, array element, or object field
- do one arithmetic or logical operation
- method call (not counting arg evaluation and execution of method body)

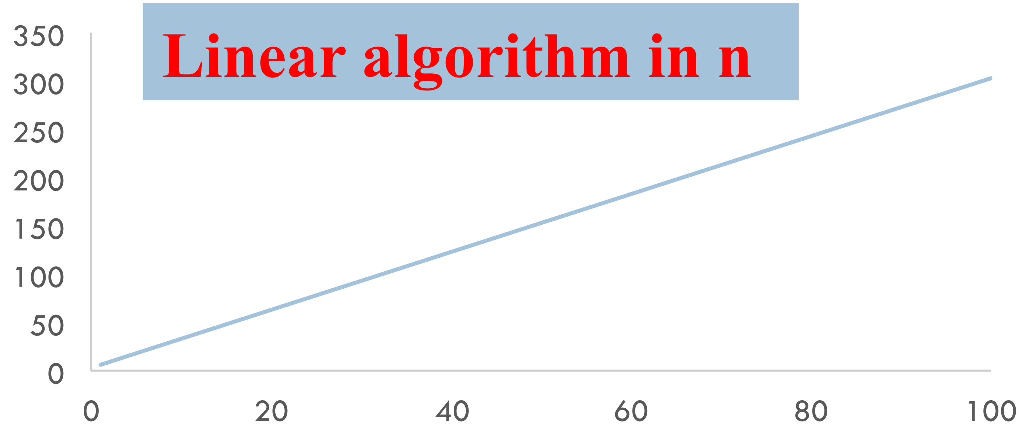
Counting Steps


```
// Store sum of 1..n in sum
sum= 0;
// inv: sum = sum of 1..(k-1)
for (int k= 1; k <= n; k= k+1){
   sum= sum + k;
}
```

Statement:	# times done
sum= 0;	1
k= 1;	1
k <= n	n+1
k= k+1;	n
sum= sum + k;	n
Total steps:	3n + 3

All basic steps take time 1.
There are n loop iterations.
Therefore, takes time proportional to n.

Linear algorithm in n



Not all operations are basic steps

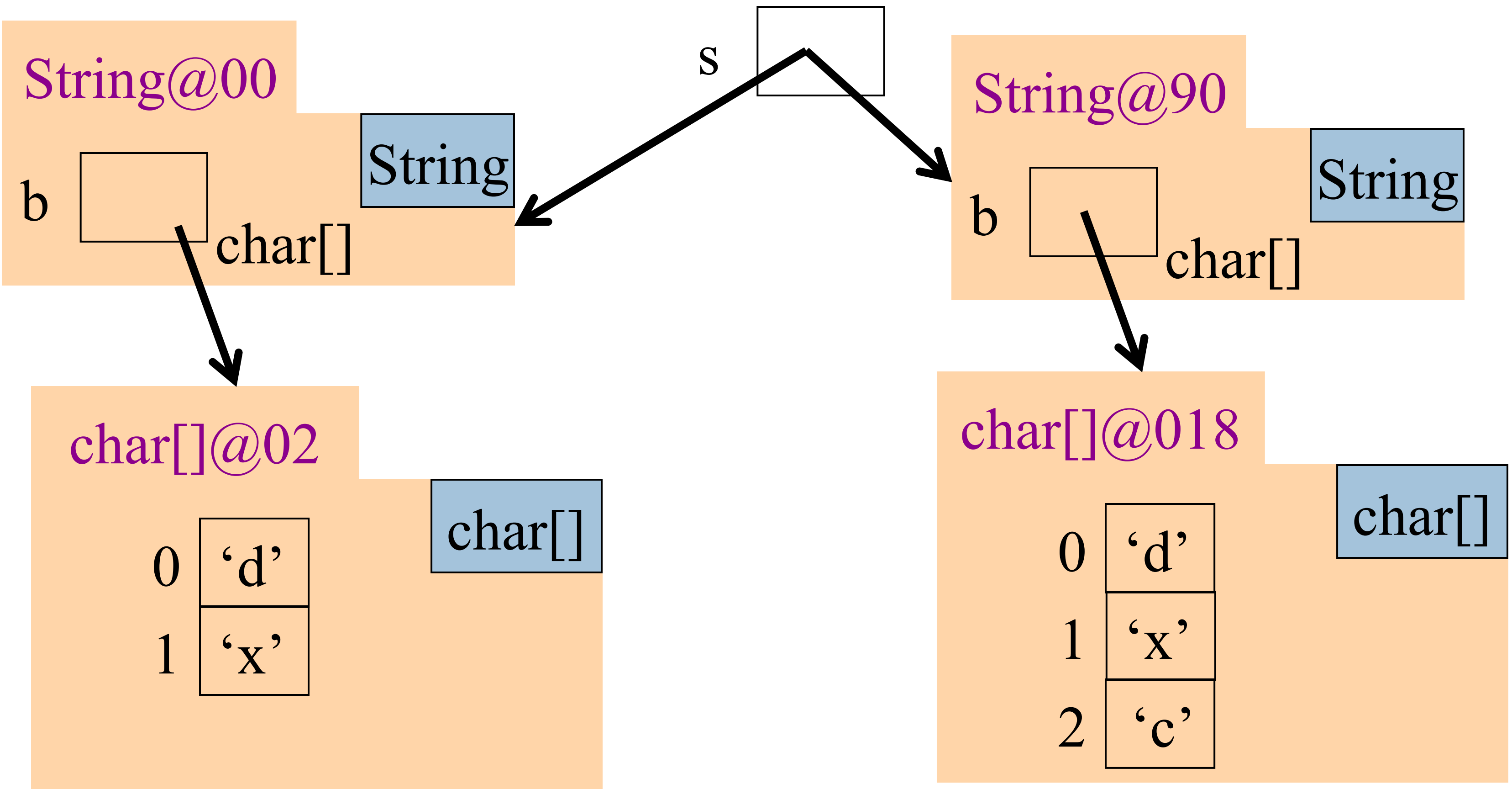

```
// Store n copies of ‘c’ in s
s= "";
// inv: s contains k-1 copies of ‘c’
for (int k= 1; k <= n; k= k+1){
    s=  s + 'c';
}
```

Statement:	# times done
s= "";	
k= 1;	1
k <= n	n+1
k= k+1;	n
s= s + 'c	
Total s	+ 3

Concatenation is not a basic step. For each k, catenation creates and fills k array elements.

String Concatenation

s= s + “c”;   is NOT constant time.
It takes time proportional to 1 + length of s

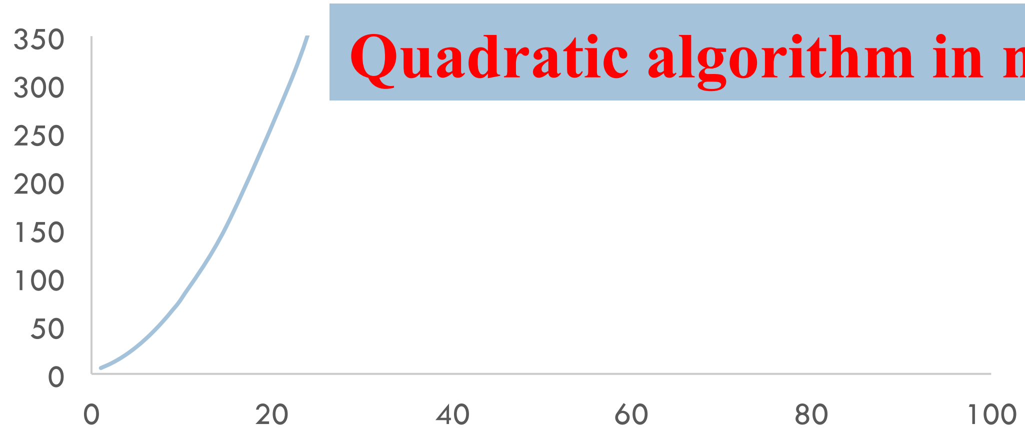


Not all operations are basic steps


```
// Store n copies of 'c' in s
s= "";
// inv: s contains k-1 copies of 'c'
for (int k= 1; k <= n; k= k+1){
    s=  s + 'c';
}
```

Statement:	# times	# steps
s= "";	1	1
k= 1;	1	1
k <= n	n+1	1
k= k+1;	n	1
s= s + 'c';	n	k
Total steps:	$n*(n-1)/2 + 2n + 3$	

Concatenation is not a basic step. For each k, catenation creates and fills k array elements.



Quadratic algorithm in n

Linear versus quadractic


```
// Store sum of 1..n in sum
sum= 0;
// inv: sum = sum of 1..(k-1)
for (int k= 1; k <= n; k= k+1)
    sum= sum + n
```

Linear algorithm

```
// Store n copies of ‘c’ in s
s= “”;
// inv: s contains k-1 copies of ‘c’
for (int k= 1; k <= n; k= k+1)
    s=  s + ‘c’;
```

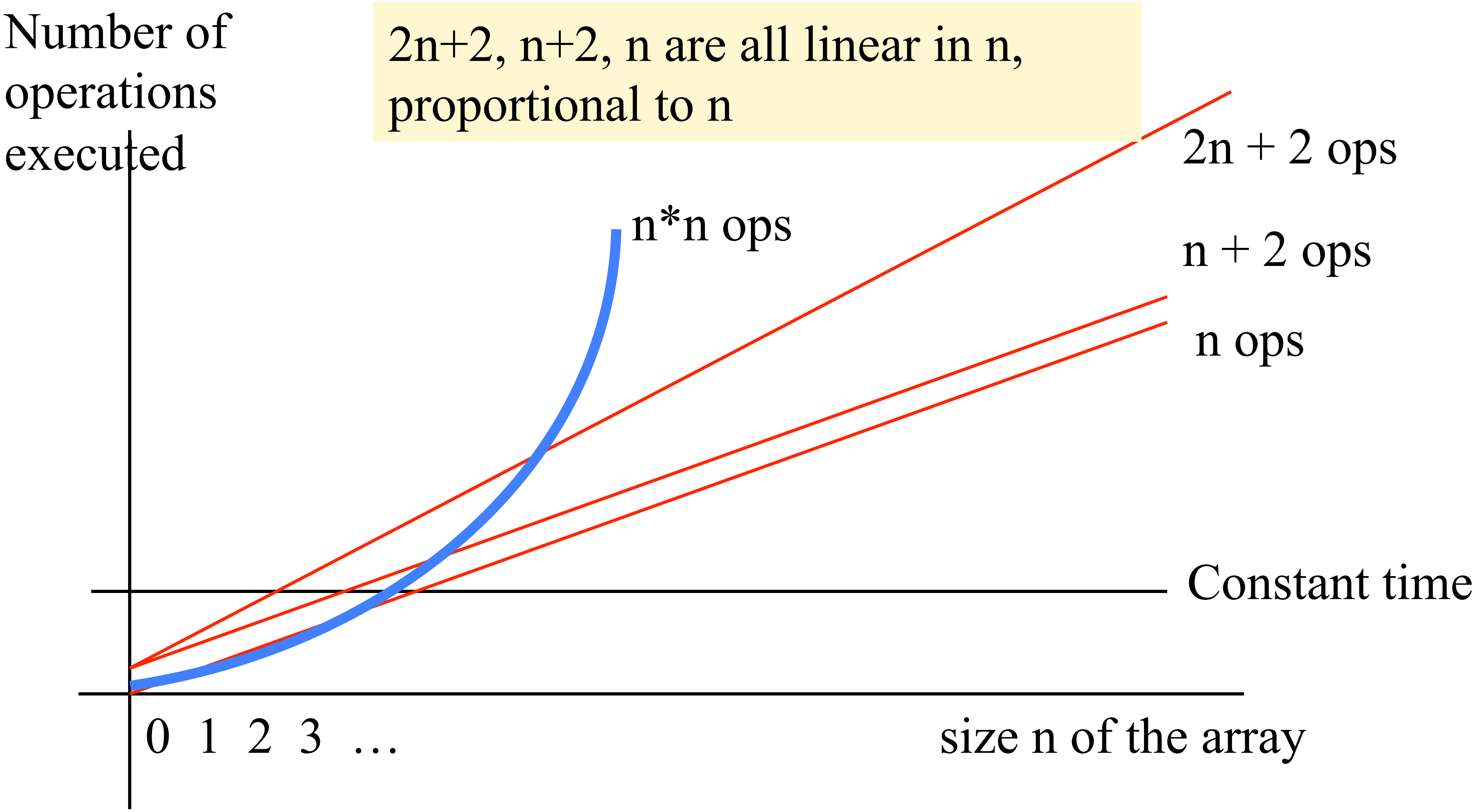
Quadratic algorithm

In comparing the runtimes of these algorithms, the exact number of basic steps is not important. What’s important is that

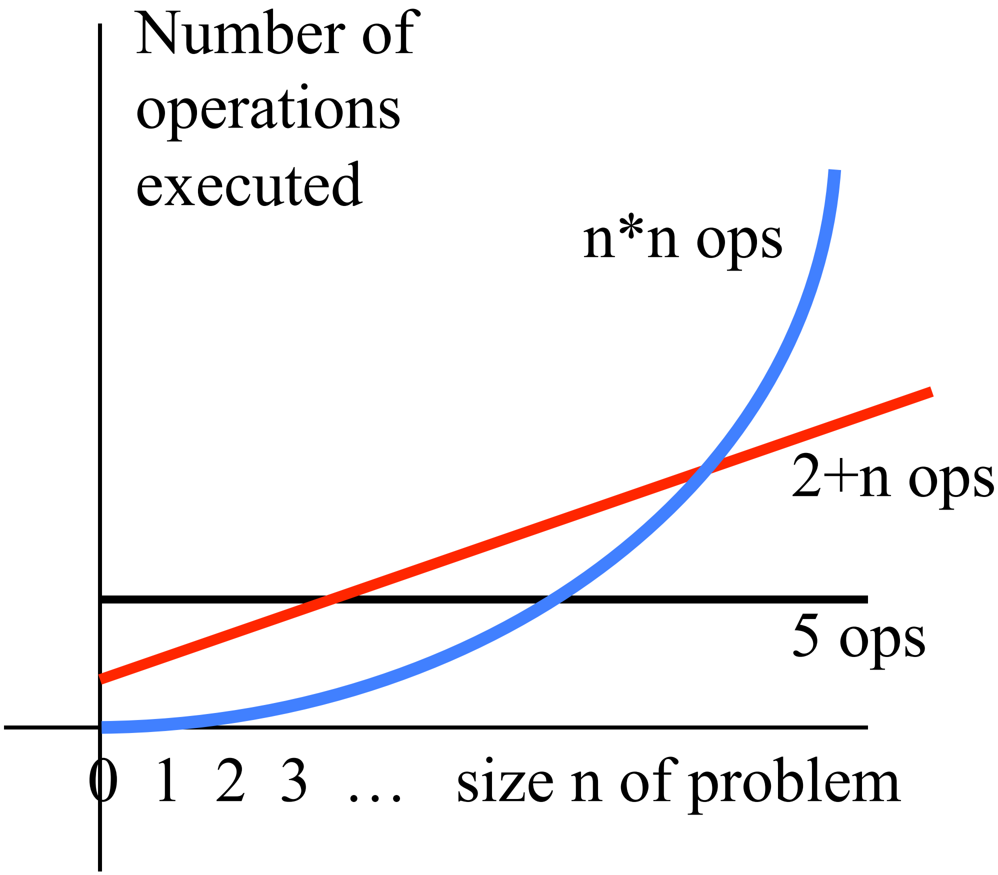
One is linear in n—takes time proportional to n

One is quadratic in n—takes time proportional to n^2

Looking at execution speed



What do we want from a definition of "runtime complexity"?



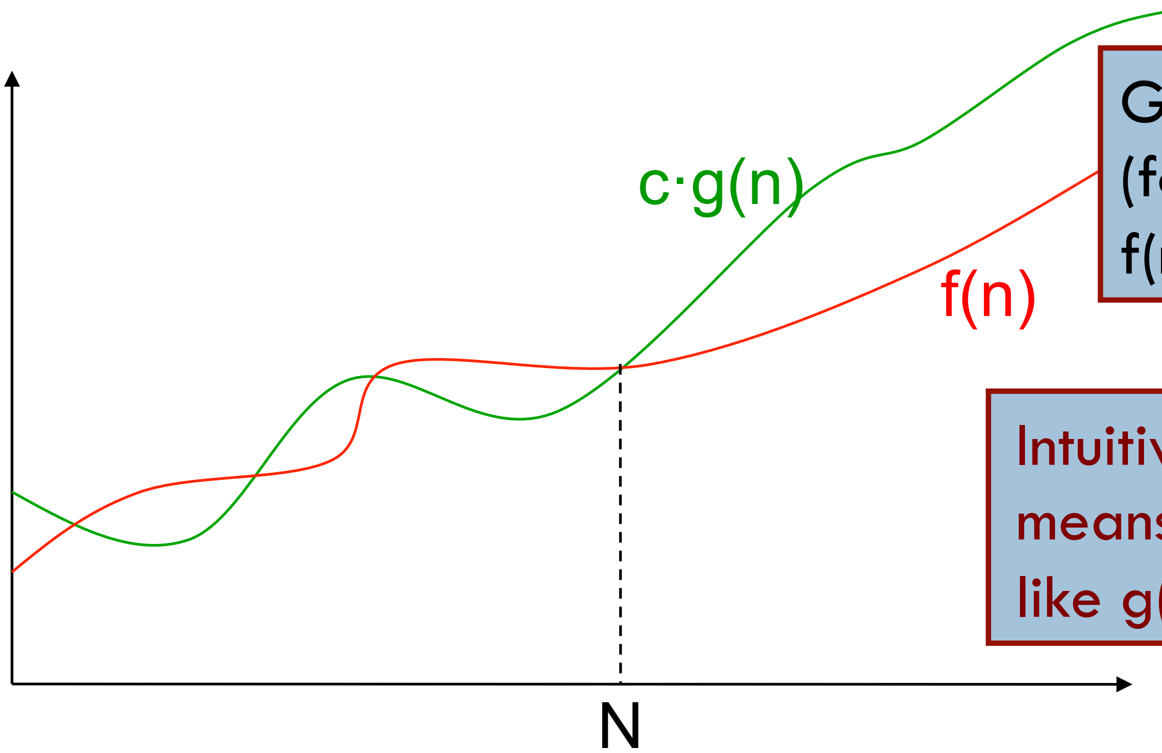
1. Distinguish among cases for large n, not small n

2. Distinguish among important cases, like
- n*n basic operations
- n basic operations
- log n basic operations
- 5 basic operations

3. Don't distinguish among trivially different cases.
- •5 or 50 operations
- •n, n+2, or 4n operations

"Big O" Notation

Formal definition: f(n) is O(g(n)) if there exist constants $c > 0$ and $N \geq 0$ such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$



Get out far enough (for $n \geq N$) f(n) is at most c·g(n)

Intuitively, f(n) is O(g(n)) means that f(n) grows like g(n) or slower

Prove that $(n^2 + n)$ is $O(n^2)$

Formal definition: $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $N \geq 0$ such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

Example: Prove that $(2n^2 + n)$ is $O(n^2)$

Methodology:

Start with $f(n)$ and slowly transform into $c \cdot g(n)$:

- Use = and <= and < steps
- At appropriate point, can choose N to help calculation
- At appropriate point, can choose c to help calculation

Prove that $(n^2 + n)$ is $O(n^2)$

Formal definition: $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $N \geq 0$ such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

Example: Prove that $(2n^2 + n)$ is $O(n^2)$

$f(n)$

= <definition of f(n)>

$2n^2 + n$

<= <for $n \geq 1$, $n \leq n^2$>

$2n^2 + n^2$

= <arith>

$3*n^2$

= <definition of $g(n) = n^2$>

$3*g(n)$

Transform $f(n)$ into $c \cdot g(n)$:

- Use =, <= , < steps
- Choose N to help calc.
- Choose c to help calc

Choose
$N = 1$ and $c = 3$

Prove that $100\ n + \log n$ is $O(n)$

Formal definition: $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $N \geq 0$ such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

$f(n)$

$=$ <put in what f(n) is>

$100\ n + \log n$

$<=$ <We know $\log n \leq n$ for $n \geq 1$>

$100\ n + n$

$=$ <arith>

$101\ n$

$=$ <$g(n) = n$>

$101\ g(n)$

Choose
$N = 1$ and $c = 101$

O(…) Examples

Let $f(n) = 3n^2 + 6n - 7$

- $f(n)$ is $O(n^2)$
- $f(n)$ is $O(n^3)$
- $f(n)$ is $O(n^4)$
- …

$p(n) = 4\ n \log n + 34\ n - 89$

- $p(n)$ is $O(n \log n)$
- $p(n)$ is $O(n^2)$

$h(n) = 20 \cdot 2^n + 40n$

$h(n)$ is $O(2^n)$

$a(n) = 34$

- $a(n)$ is $O(1)$

Only the *leading* term (the term that grows most rapidly) matters

If it's $O(n^2)$, it's also $O(n^3)$ etc! However, we always use the smallest one

Do NOT say or write $f(n) = O(g(n))$

Formal definition: $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $N \geq 0$ such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

$f(n) = O(g(n))$ is simply WRONG. Mathematically, it is a disaster. You see it sometimes, even in textbooks. Don't read such things.

Here's an example to show what happens when we use = this way.

We know that $n+2$ is $O(n)$ and $n+3$ is $O(n)$. Suppose we use =

$$n+2 = O(n)$$
$$n+3 = O(n)$$

But then, by transitivity of equality, we have $n+2 = n+3$.
We have proved something that is false. Not good.

Problem-size examples

- Suppose a computer can execute 1000 operations per second; how large a problem can we solve?

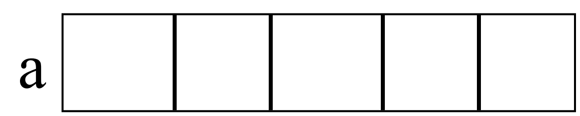
operations	1 second	1 minute	1 hour
n	1000	60,000	3,600,000
n log n	140	4893	200,000
n^2	31	244	1897
$3n^2$	18	144	1096
n^3	10	39	153
2^n	9	15	21

Commonly Seen Time Bounds

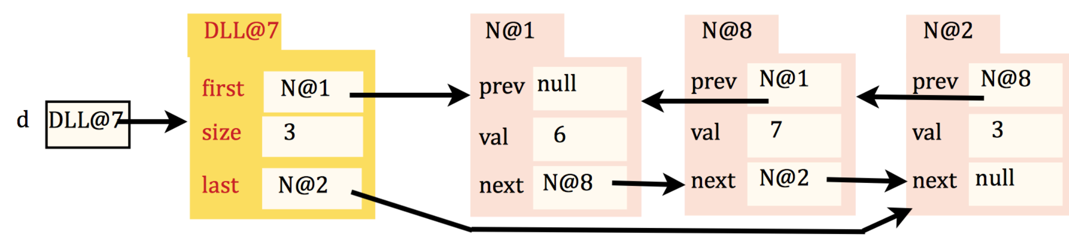
$O(1)$	constant	excellent
$O(\log n)$	logarithmic	excellent
$O(n)$	linear	good
$O(n \log n)$	n log n	pretty good
$O(n^2)$	quadratic	maybe OK
$O(n^3)$	cubic	maybe OK
$O(2^n)$	exponential	too slow

Big O Poll

Consider two different data structures that could store your data: an array or a doubly-linked list. In both cases, let n be the size of your data structure (i.e., the number of elements it is currently storing). What is the running time of each of the following operations:



- get(i) using an array
- get(i) using a DLL
- insert(v) using an array
- insert(v) using a DLL

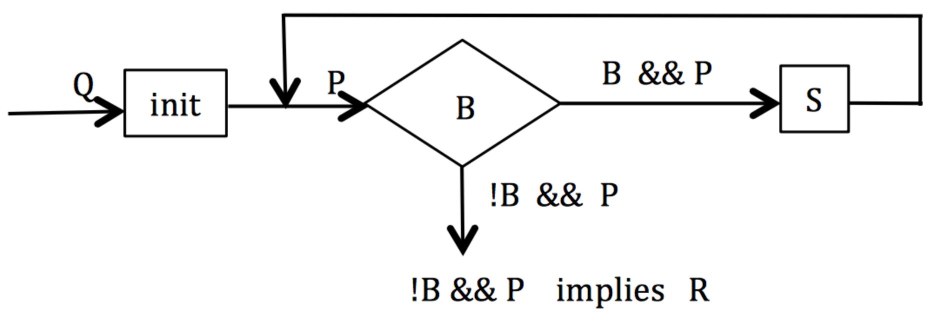


Java Lists

- java.util defines an interface List<E>
- implemented by multiple classes:
  - ArrayList
  - LinkedList

Search for v in b[0..]

// Store in i the index of the first occurrence of v in array b
// Precondition: v is in b.



Methodology:

1. Define pre and post conditions.
2. Draw the invariant as a combination of pre and post.
3. Develop loop using 4 loopy questions.

Practice doing this!

Search for v in b[0..]

// Store in i the index of the first occurrence of v in array b
// Precondition: v is in b.

	0			b.length
pre: b	v in here			

	0	i		b.length
post: b	!= v	v	?	

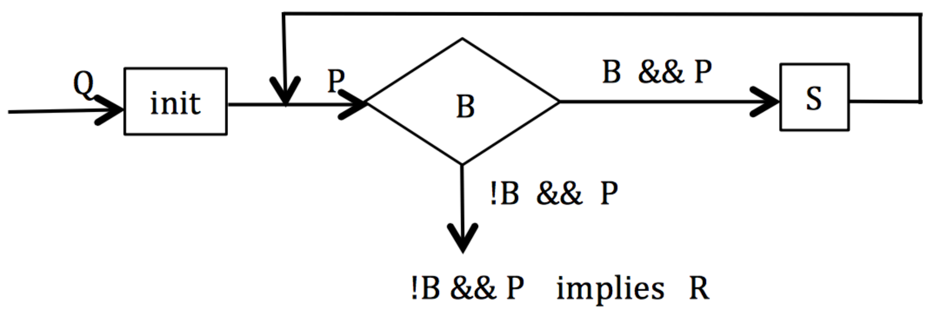
	0	i		b.length
inv: b	!= v	v in here		

Methodology:

1. Define pre and post conditions.
2. Draw the invariant as a combination of pre and post.
3. Develop loop using 4 loopy questions.

Practice doing this!

The Four Loopy Questions



- Does it start right?

  Is {Q} init {P} true?

- Does it continue right?

  Is {P && B} S {P} true?

- Does it end right?

  Is P && !B => R true?

- Will it get to the end?

  Does it make progress toward termination?

Search for v in b[0..]

// Store in i the index of the first occurrence of v in array b

// Precondition: v is in b.

	0			b.length
pre: b	v in here			

	0	i		b.length
post: b	!= v	v	?	

	0	i		b.length
inv: b	!= v	v in here		

```
i= 0;
while (b[i] != v) {
    i= i+1;
}
```

Each iteration takes constant time.

Worst case: b.length-1 iterations

Linear algorithm: O(b.length)

Search for v in sorted b[0..]

// Store in i to truthify b[0..i] <= v < b[i+1..]

// Precondition: b is sorted.

pre: b

0	b.length
sorted	

post: b

0	i	b.length
<= v	> v	

inv: b

0	i	k	b.length
<= v	sorted	> v	

Methodology:

1. Define pre and post conditions.
2. Draw the invariant as a combination of pre and post.
3. Develop loop using 4 loopy questions.

Practice doing this!

Another way to search for v in b[0..]

// Store in i to truthify b[0..i] <= v < b[i..]

// Precondition: b is sorted.

pre: b — 0 … b.length: | sorted |

post: b — 0, i, b.length: | <= v | > v |

inv: b — 0, i, k, b.length: | <= v | sorted | > v |

b — 0, i, j, k, b.length: | <= v | | | | > v |

j = (i+k)/2

```
i= -1;
k= b.length;
while  (i < k-1) {
    int j=  (i+k)/2;
    // i < j < k
    if (b[j] <= v)  i=  j;
    else  k=  j;
}
```

Another way to search for v in b[0..]

// Store in i to truthify b[0..i] <= v < b[i..]

// Precondition: b is sorted.

pre: b — 0 … b.length: | sorted |

post: b — 0 … i … b.length: | <= v | > v |

inv: b — 0 … i … k … b.length: | <= v | sorted | > v |

```
i= -1;
k= b.length;
while (i < k-1) {
    int j= (i+k)/2;
    // i < j < k
    if (b[j] <= v) i= j;
    else k= j;
}
```

Each iteration takes constant time.

Logarithmic: O(log(b.length))

Worst case: log(b.length) iterations

Another way to search for v in b[0..]

// Store in i to truthify b[0..i] <= v < b[i+1..]

// Precondition: b is sorted.

This algorithm is better than binary searches that stop when v is found.

1. Gives good info when v not in b.
2. Works when b is empty.
3. Finds last occurrence of v, not arbitrary one.
4. Correctness, including making progress, easily seen using invariant

```
i= 0;
k= b.length;
while  (i < k-1 ) {
    int j=  (i+k)/2;
    // i < j < k
    if (b[j] <= v)  i=  j;
    else  k=  j;
}
```

Logarithmic: O(log(b.length))

Dutch National Flag Algorithm

Dutch National Flag Algorithm

Dutch national flag. Swap b[0..n-1] to put the reds first, then the whites, then the blues. That is, given precondition Q, swap values of b[0.n] to truthify postcondition R:

	0		n
Q: b	?		

	0			n
R: b	reds	whites	blues	

	0				n
P1: b	reds	whites	blues	?	

	0				n
P2: b	reds	whites	?	blues	

Dutch National Flag Algorithm: invariant P1

0 n

Q: b | ? |

0 n

R: b | reds | whites | blues |

0 h k p n

P1: b | reds | whites | blues | ? |

```
h= 0; k= h; p= k;
while ( p != n ) {
    if (b[p] blue)  p= p+1;
    else if (b[p] white) {
        swap b[p], b[k];
        p= p+1; k= k+1;
    }
    else { // b[p] red
        swap b[p], b[h];
        swap b[p], b[k];
        p= p+1; h=h+1; k= k+1;
    }
}
```

Dutch National Flag Algorithm: invariant P2

	0			n
Q: b	?			

	0			n
R: b	reds	whites	blues	

	0	h	k	p	n
P2: b	reds	whites	?	blues	

```
h= 0; k= h; p= n;
while ( k != p ) {
    if (b[k] white)  k= k+1;
    else if (b[k] blue) {
        p= p-1;
        swap b[k], b[p];
    }
    else { // b[k] is red
        swap b[k], b[h];
        h= h+1; k= k+1;
    }
}
```

Asymptotically, which algorithm is faster?

Invariant 1

0	h	k	p	n
reds	whites	blues	?	

```
h= 0; k= h; p= k;
while ( p != n ) {
    if (b[p] blue)        p=  p+1;
    else if (b[p] white) {
            swap b[p], b[k];
            p= p+1; k= k+1;
    }
    else { // b[p] red
            swap b[p], b[h];
            swap b[p], b[k];
            p= p+1; h=h+1; k= k+1;
    }
}
```

Invariant 2

0	h	k	p	n
reds	whites	?	blues	

```
h= 0; k= h; p= n;
while ( k != p ) {
    if (b[k] white)    k=  k+1;
    else if (b[k] blue) {
            p= p-1;
            swap b[k], b[p];
    }
    else { // b[k] is red
            swap b[k], b[h];
            h= h+1; k= k+1;
    }
}
```

Asymptotically, which algorithm is faster?

Invariant 1

0	h	k	p	n
reds	whites	blues	?	

might use 2 swaps per iteration

```
    if (b[p] blue)          p=  p+1;
    else if (b[p] white) {
            swap b[p], b[k];
```

```
            swap b[p], b[h];
            swap b[p], b[k];
            p= p+1; h=h+1; k= k+1;
    }
}
```

Invariant 2

0	h	k	p	n
reds	whites	?	blues	

uses at most 1 swap per iteration

```
    if (b[k] white)          k=  k+1;
    else if (b[k] blue) {
            p= p-1;
```

```
            swap b[k], b[h];
            h= h+1; k= k+1;
    }
}
```

These two algorithms have the same asymptotic running time (both are O(n))