RECURSION

Lecture 6

CS2110 – Spring 2013

Recursion

- Arises in three forms in computer science
  - Recursion as a *mathematical* tool for defining a function in terms of its own value in a simpler case
  - Recursion as a *programming* tool. You've seen this previously but we'll take it to mind-bending extremes (by the end of the class it will seem easy!)
  - Recursion used to prove properties about algorithms. We use the term *induction* for this and will discuss it later.

Recursion as a math technique

- Broadly, recursion is a powerful technique for specifying functions, sets, and programs
- A few recursively-defined functions and programs
    - factorial
    - combinations
    - exponentiation (raising to an integer power)
- Some recursively-defined sets
    - grammars
    - expressions
    - data structures (lists, trees, ...)

Example: Sum the digits in a number

```
/** return sum of digits in n, given n >= 0 */
 public static int sum(int n) {
      if (n < 10) return n;

     // n has at least two digits:
     // return first digit + sum of rest
     return n%10 + sum(n/10);
 }
```

sum calls itself!

- E.g. sum(87012) = 2+(1+(0+(7+8))) = 18

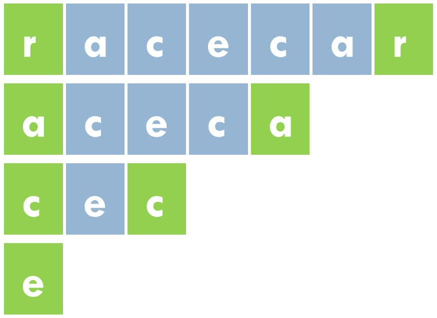
Example: Is a string a palindrome?

```
/** = "s is a palindrome" */
  public static boolean isPalindrome(String s) {
     if (s.length() <= 1)
        return true;

     // s has at least 2 chars
     int n= s.length()-1;
     return s.charAt(0) == s.charAt(n) && isPalindrome(s.substring(1, n));
  }
```

Substring from char(1) to char(n-1)

- isPalindrome(“racecar”) = true
- isPalindrome(“pumpkin”) = false


r a c e c a r
a c e c a
c e c
e

Count the e’s in a string

```
/** = " number of times c occurs in s */
   public static int countEm(char c, String s) {
       if (s.length() == 0)
           return 0;

       // { s has at least 1 character }
       if (s.charAt(0) != c)
           return countEm(c, s.substring(1));

       // { first character of s is c}
       return 1 + countEm (c, s.substring(1));
   }
```

Substring from char(1) to end

- **countEm(‘e’, “it is easy to see that this has many e’s”) = 4**
- **countEm(‘e’, “Mississippi”) = 0**

The Factorial Function (n!)

- Define n! = n·(n–1)·(n–2)···3·2·1

  read: "n factorial"

  - E.g., 3! = 3·2·1 = 6

- By convention, 0! = 1

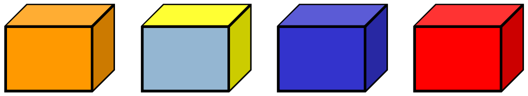
- The function int → int that gives n! on input n is called the factorial function

The Factorial Function (n!)

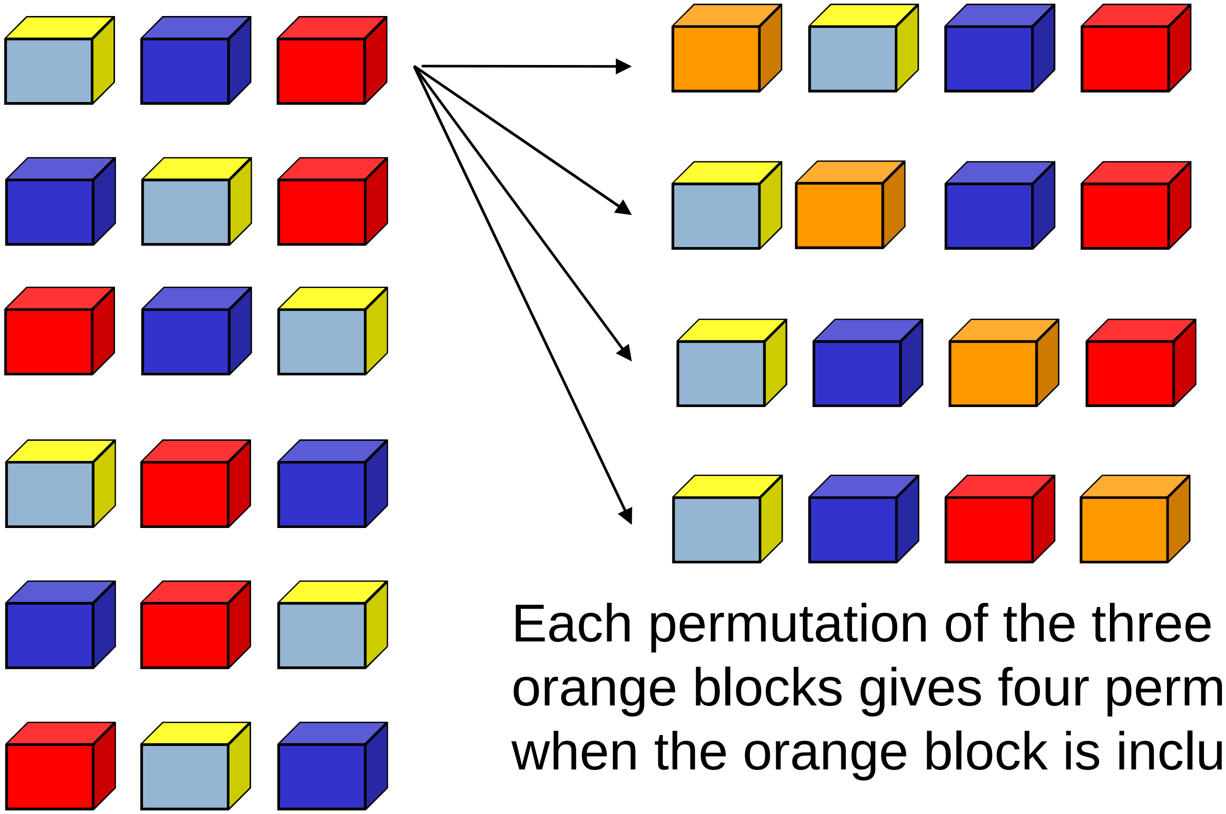
- n! is the number of permutations of n distinct objects
  - There is just one permutation of one object. 1! = 1
  - There are two permutations of two objects: 2! = 2

    1 2   2 1
  - There are six permutations of three objects: 3! = 6

    1 2 3   1 3 2   2 1 3   2 3 1   3 1 2   3 2 1
- If $n > 0$, $n! = n\cdot(n-1)!$

Permutations of 

Permutations of
non-orange blocks



Each permutation of the three non-orange blocks gives four permutations when the orange block is included

- Total number = 4·3! = 4·6 = 24: 4!

Observation

- One way to think about the task of permuting the four colored blocks was to start by computing all permutations of three blocks, then finding all ways to add a fourth block
  - And this “explains” why the number of permutations turns out to be 4!
  - Can generalize to prove that the number of permutations of n blocks is n!

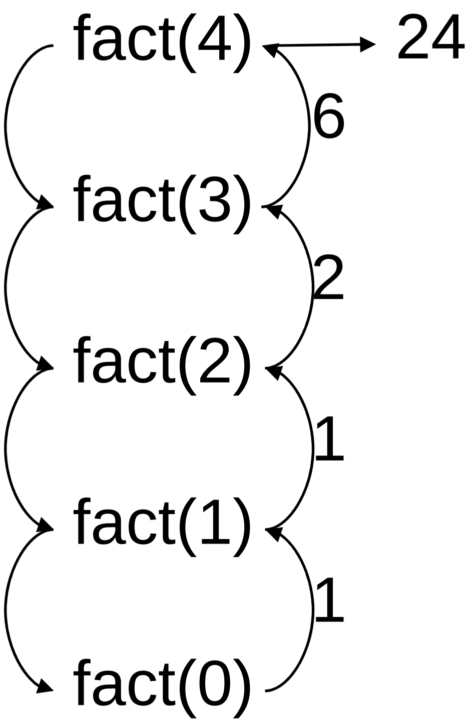
A Recursive Program

$0! = 1$

$n! = n \cdot (n-1)!,\quad n > 0$

```
static int fact(int n) {
   if (n == 0)
        return 1;
   else
        return n*fact(n-1);
}
```

Execution of fact(4)



fact(4)

24

6

fact(3)

2

fact(2)

1

fact(1)

1

fact(0)

General Approach to Writing Recursive Functions

1. Try to find a parameter, say n, such that the solution for n can be obtained by combining solutions to the *same problem using smaller values of n* (e.g., (n-1) in our factorial example)

2. Find *base case(s)* – small values of n for which you can just write down the solution (e.g., 0! = 1)

3. Verify that, for any valid value of n, applying the reduction of step 1 repeatedly will ultimately hit one of the base cases

A cautionary note

- Keep in mind that each instance of your recursive function has its own local variables
- Also, remember that “higher” instances are waiting while “lower” instances run

- Not such a good idea to touch global variables from within recursive functions
  - Legal… but a common source of errors
  - Must have a really clear mental picture of how recursion is performed to get this right!

The Fibonacci Function

- **Mathematical definition:**

  fib(0) = 0
  fib(1) = 1
  fib(n) = fib(n – 1) + fib(n – 2), n ≥ 2

  two base cases!

- **Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, …**

```
static int fib(int n) {
   if (n == 0) return 0;
   else if (n == 1) return 1;
   else return fib(n-2) + fib(n-1);
}
```

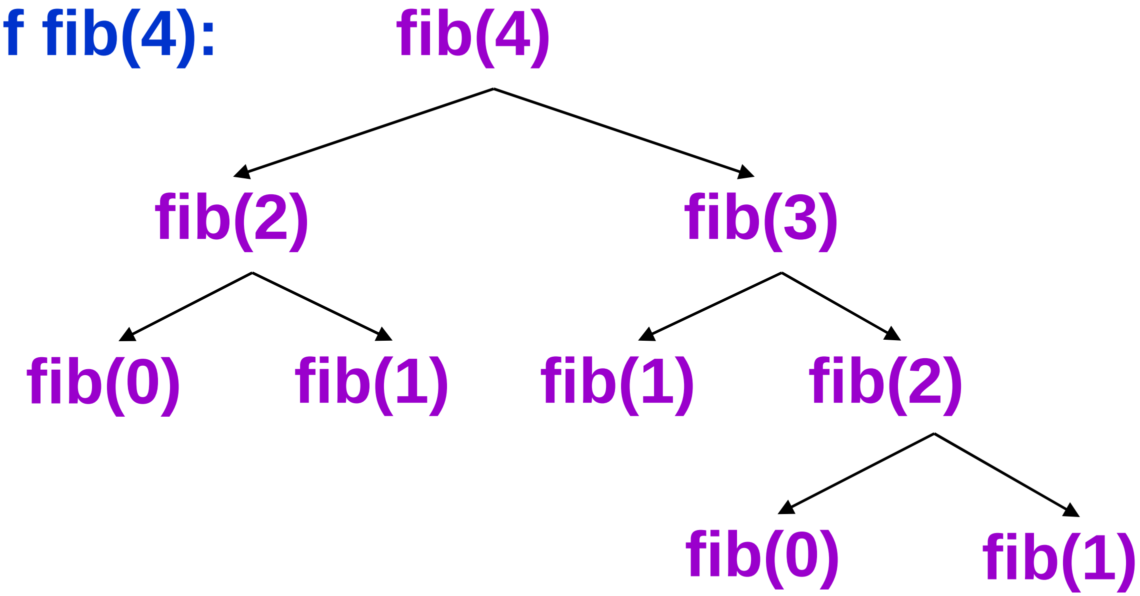
Fibonacci (Leonardo Pisano) 1170–1240?

Statue in Pisa, Italy
Giovanni Paganucci
1863

Recursive Execution

```
static int fib(int n) {
   if (n == 0) return 0;
   else if (n == 1) return 1;
   else return fib(n-2) + fib(n-1);
}
```

Execution of fib(4):



fib(4)

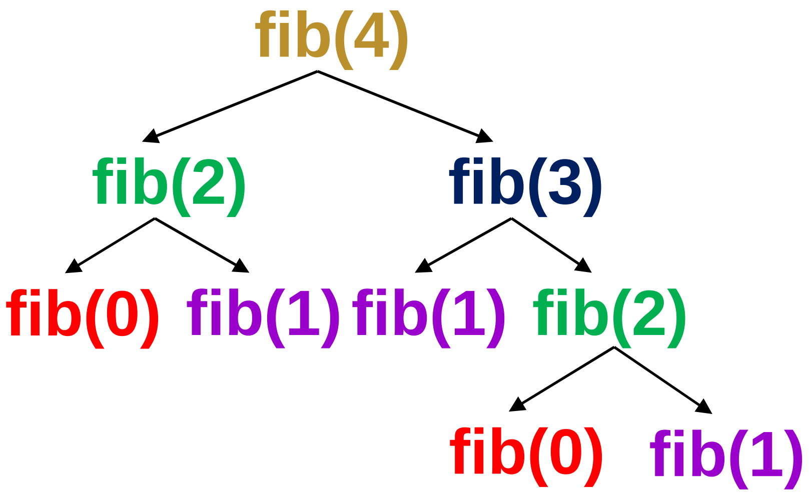
fib(2) fib(3)

fib(0) fib(1) fib(1) fib(2)

fib(0) fib(1)

One thing to notice

- This way of computing the Fibonacci function is elegant, but inefficient
- It “recomputes” answers again and again!
- To improve speed, need to save known answers in a table!
  - One entry per answer
  - Such a table is called a *cache*



fib(4)

fib(2) fib(3)

fib(0) fib(1) fib(1) fib(2)

fib(0) fib(1)

Memoization (fancy term for “caching”)

- Memoization is an optimization technique used to speed up computer programs by having function calls avoid repeating the calculation of results for previously processed inputs.
  - The first time the function is called, we save result
  - The next time, we can look the result up
    - Assumes a “side effect free” function: The function just computes the result, it doesn’t change things
    - If the function depends on anything that changes, must “empty” the saved results list

Adding Memoization to our solution

- Before:

```
static i
   if (n
       r
   else
       r
   else
       r
}
```

- After

```
static ArrayList<Integer> cached =
                       new ArrayList<Integer>();

static int fib(int n) {
   if(n < cached.size())
       return cached.get(n);
   int v;
   if (n == 0)
       v = 0;
   else if (n == 1)
       v = 1;
   else
       v = fib(n-2) + fib(n-1);
   // cached[n] = fib(n).  This code makes use of the fact
   // that an ArrayList adds elements to the end of the list
   if(n == cached.size())
       cached.add(v);
   return v;
}
```

Notice the development process

- We started with the idea of recursion
- Created a very simple recursive procedure
- Noticed it will be slow, because it wastefully recomputes the same thing again and again
- So made it a bit more complex but gained a lot of speed in doing so

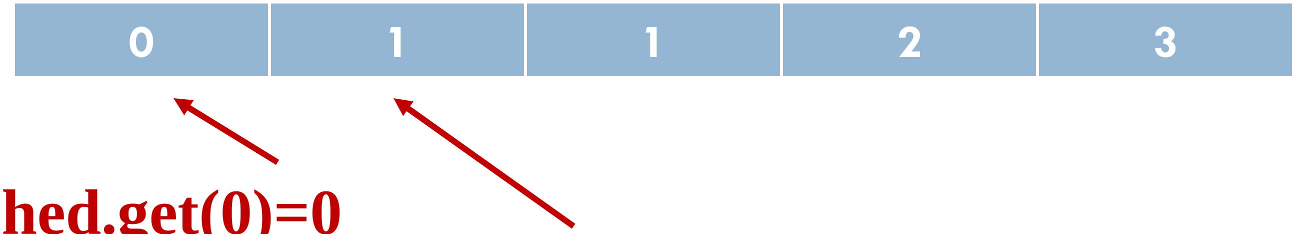
- This is a common software engineering pattern

Why did it work?

- This cached list “works” because for each value of n, either cached.get(n) is still undefined, or has fib(n)

- Takes advantage of the fact that an ArrayList adds elements at the end, and indexes from 0

cached@BA8900, size=5



cached.get(0)=0

cached.get(1)=1

… cached.get(n)=fib(n)

Property of our code: cached.get(n) accessed <u>after</u> fib(n) computed

Positive Integer Powers

- $a^n = a \cdot a \cdot a \cdots a$ (n times)

- Alternate description:
  - $a^0 = 1$
  - $a^{n+1} = a \cdot a^n$

```
static int power(int a, int n) {
   if (n == 0) return 1;
   else return a*power(a,n-1);
}
```

A Smarter Version

- Power computation:
  - a0 = 1
  - If n is nonzero and even, an = (an/2)2
  - If n is odd, an = a·(an/2)2
    - Java note: If x and y are integers, “x/y” returns the integer part of the quotient
- Example:
  - a5 = a·(a5/2)2 = a·(a2)2 = a·((a2/2)2)2 = a·(a2)2

Note: this requires 3 multiplications rather than 5!

A Smarter Version

- … Example:
  - a5 = a·(a5/2)2 = a·(a2)2 = a·((a2/2)2)2 = a·(a2)2

    Note: this requires 3 multiplications rather than 5!

- What if n were larger?
  - Savings would be more significant
- This is much faster than the straightforward computation
  - Straightforward computation: n multiplications
  - Smarter computation: log(n) multiplications

Smarter Version in Java

- n = 0: $a^0 = 1$
- n nonzero and even: $a^n = (a^{n/2})^2$
- n nonzero and odd: $a^n = a\cdot(a^{n/2})^2$

local variable

parameters

```
static int power(int a, int n) {
   if (n == 0) return 1;
   int halfPower = power(a,n/2);
   if (n%2 == 0) return halfPower*halfPower;
   return halfPower*halfPower*a;
}
```

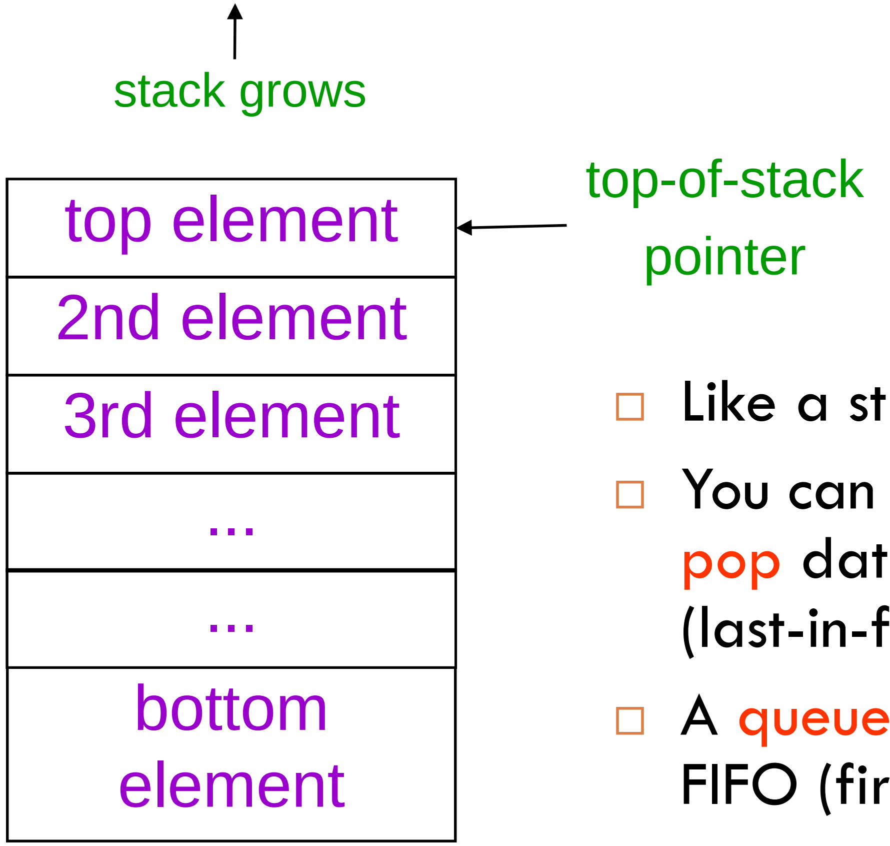
- The method has two parameters and a local variable
- Why aren’t these overwritten on recursive calls?

How Java “compiles” recursive code

- Key idea:
  - Java uses a stack to remember parameters and local variables across recursive calls
  - Each method invocation gets its own stack frame

- A stack frame contains storage for
  - Local variables of method
  - Parameters of method
  - Return info (return address and return value)
  - Perhaps other bookkeeping info

Stacks



stack grows

top-of-stack pointer

top element

2nd element

3rd element

...

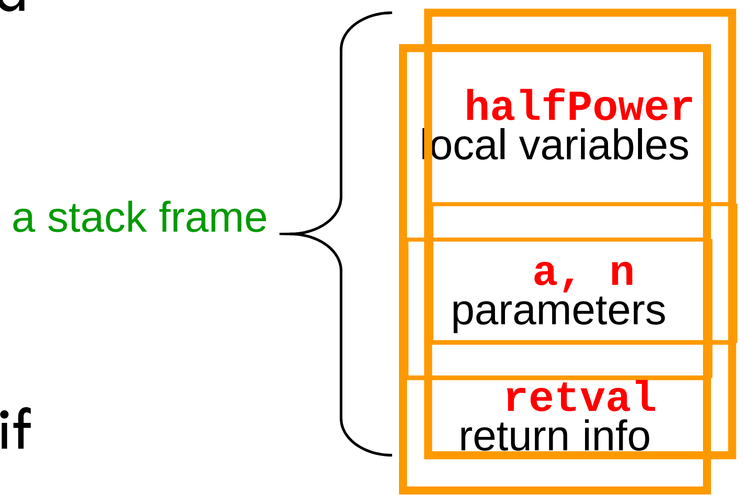
...

bottom element

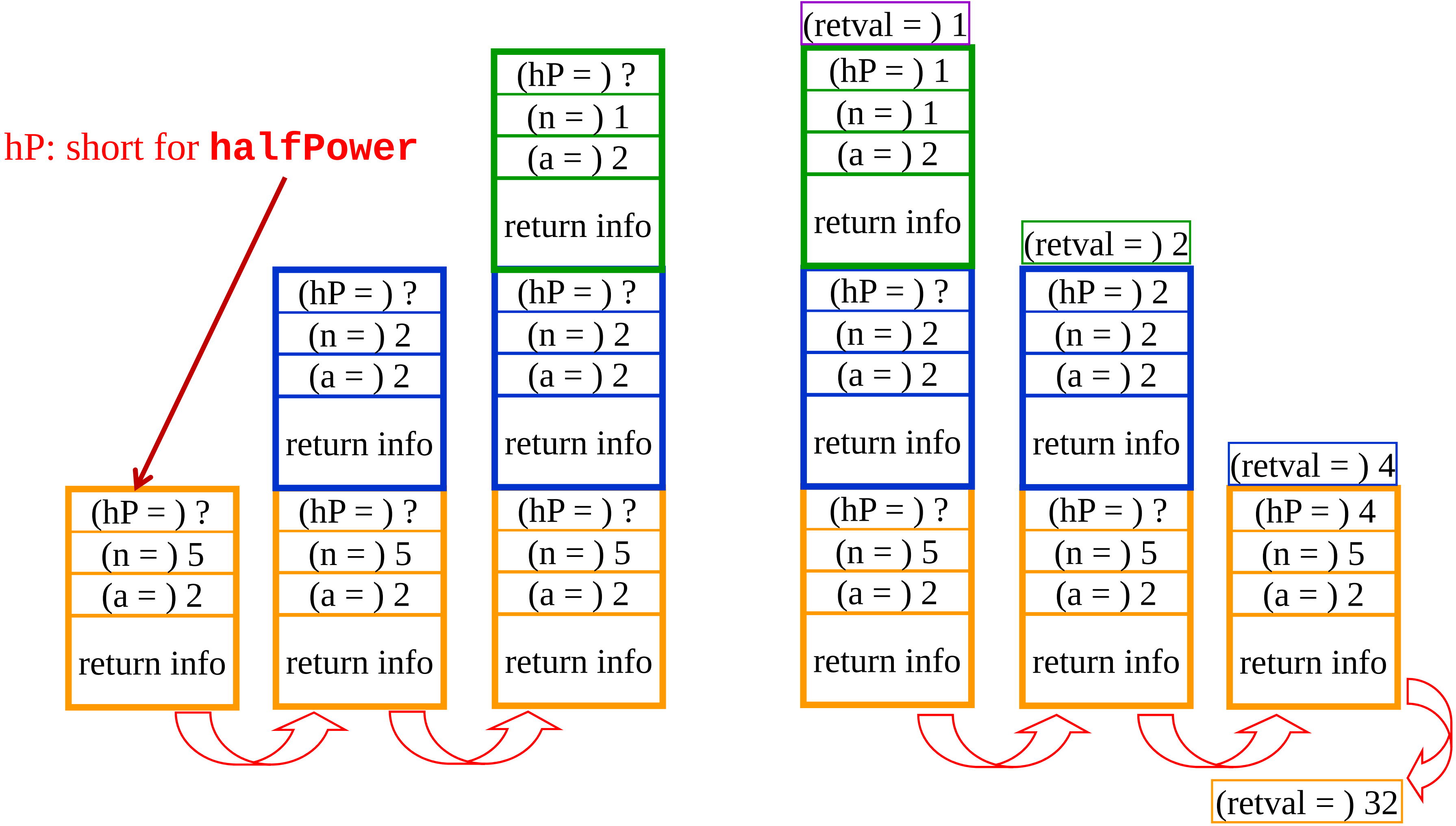
- Like a stack of dinner plates
- You can push data on top or pop data off the top in a LIFO (last-in-first-out) fashion
- A queue is similar, except it is FIFO (first-in-first-out)

Stack Frame

- A new stack frame is pushed with each recursive call
- The stack frame is popped when the method returns
  - Leaving a return value (if there is one) on top of the stack



Example: power(2, 5)



hP: short for `halfPower`

(hP =) ?
(n =) 5
(a =) 2
return info

(hP =) ?
(n =) 2
(a =) 2
return info
(hP =) ?
(n =) 5
(a =) 2
return info

(hP =) ?
(n =) 1
(a =) 2
return info
(hP =) ?
(n =) 2
(a =) 2
return info
(hP =) ?
(n =) 5
(a =) 2
return info

(retval =) 1
(hP =) 1
(n =) 1
(a =) 2
return info
(hP =) ?
(n =) 2
(a =) 2
return info
(hP =) ?
(n =) 5
(a =) 2
return info

(retval =) 2
(hP =) 2
(n =) 2
(a =) 2
return info
(hP =) ?
(n =) 5
(a =) 2
return info

(retval =) 4
(hP =) 4
(n =) 5
(a =) 2
return info

(retval =) 32

How Do We Keep Track?

- Many frames may exist, but computation is only occurring in the top frame
    - The ones below it are waiting for results

- The hardware has nice support for this way of implementing function calls, and recursion is just a kind of function call

Conclusion

- Recursion is a convenient and powerful way to define functions
- Problems that seem insurmountable can often be solved in a "divide-and-conquer" fashion:
  - Reduce a big problem to smaller problems of the same kind, solve the smaller problems
  - Recombine the solutions to smaller problems to form solution for big problem
- Important application (next lecture): parsing

Extra slides

- For use if we have time for one more example of recursion

- This builds on the ideas in the Fibonacci example

Combinations
(a.k.a. Binomial Coefficients)

- How many ways can you choose r items from a set of n distinct elements? $\binom{n}{r}$ **"n choose r"**

$\binom{5}{2}$ = number of 2-element subsets of {A,B,C,D,E}

2-element subsets containing A: $\binom{4}{1}$
{A,B}, {A,C}, {A,D}, {A,E}

2-element subsets not containing A: {B,C},{B,D},{B,E},{C,D},{C,E},{D,E}

$$\binom{4}{2}$$

- Therefore, $\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$
- … in perfect form to write a recursive function!

Combinations

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$

$$\binom{n}{n} = 1$$

$$\binom{n}{0} = 1$$

Can also show that $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

Pascal's triangle

=

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Binomial Coefficients

Combinations are also called *binomial coefficients* because they appear as coefficients in the expansion of the binomial power **$(x+y)^n$** :

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i$$

Combinations Have Two Base Cases

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$

$$\binom{n}{n} = 1$$

$$\binom{n}{0} = 1$$

Two base cases

- Coming up with right base cases can be tricky!
- General idea:
  - Determine argument values for which recursive case does not apply
  - Introduce a base case for each one of these

Recursive Program for Combinations

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$

$$\binom{n}{n} = 1$$

$$\binom{n}{0} = 1$$

```
static int combs(int n, int r) {   //assume n>=r>=0
   if (r == 0 || r == n) return 1; //base cases
   else return combs(n-1,r) + combs(n-1,r-1);
}
```

Exercise for the reader (you!)

- Modify our recursive program so that it caches results
- Same idea as for our caching version of the fibonacci series

- Question to ponder: When is it worthwhile to adding caching to a recursive function?
  - *Certainly not always…*
  - *Must think about tradeoffs: space to maintain the cached results vs speedup obtained by having them*

Something to think about

- With fib(), it was kind of a trick to arrange that:

  cached[n]=fib(n)

- Caching combinatorial values will force you to store more than just the answer:
  - Create a class called `Triple`
  - Design it to have integer fields `n, r, v`
  - Store Triple objects into `ArrayList<Triple> cached`;
  - Search `cached` for a saved value matching `n` and `r`
    - Hint: use a foreach loop