Inductive proof that $\mathsf{fib}(n) = \dfrac{\varphi^n - \varphi'^n}{\sqrt{5}}$

Let $\mathsf{fib}(n)$ be the nth Fibonacci number defined by the recurrence:

$$\mathsf{fib}(0) = 0 \qquad \mathsf{fib}(1) = 1 \qquad \mathsf{fib}(n) = \mathsf{fib}(n-1) + \mathsf{fib}(n-2), \quad n \geq 2. \tag{1}$$

The Fibonacci sequence is 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

Let φ and φ' be the roots of the quadratic polynomial $x^2 - x - 1$:

$$\varphi = \frac{1+\sqrt{5}}{2} \qquad \varphi' = \frac{1-\sqrt{5}}{2}$$

The positive root φ is called the *golden ratio*. Being roots of the polynomial $x^2 - x - 1$ says that

$$\varphi^2 = \varphi + 1 \qquad \varphi'^2 = \varphi' + 1.$$

Multiplying both sides of these equations by φ^{n-2} and φ'^{n-2} respectively,

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2} \qquad \varphi'^n = \varphi'^{n-1} + \varphi'^{n-2}. \tag{2}$$

Now we proceed by induction on n.

Basis For $n = 0$ and $n = 1$,

$$\begin{aligned} \mathsf{fib}(0) = 0 &= \frac{1-1}{\sqrt{5}} = \frac{\varphi^0 - \varphi'^0}{\sqrt{5}}, \\ \mathsf{fib}(1) = 1 &= \frac{1+\sqrt{5}-(1-\sqrt{5})}{2\sqrt{5}} = \frac{\varphi^1 - \varphi'^1}{\sqrt{5}}. \end{aligned}$$

Induction Step For $n \geq 2$,

$$\begin{aligned} \mathsf{fib}(n) &= \mathsf{fib}(n-1) + \mathsf{fib}(n-2) && \text{by (1)} \\ &= \frac{\varphi^{n-1} - \varphi'^{n-1}}{\sqrt{5}} + \frac{\varphi^{n-2} - \varphi'^{n-2}}{\sqrt{5}} && \text{by the induction hypothesis} \\ &= \frac{\varphi^{n-1} + \varphi^{n-2} - (\varphi'^{n-1} + \varphi'^{n-2})}{\sqrt{5}} \\ &= \frac{\varphi^n - \varphi'^n}{\sqrt{5}} && \text{by (2).} \end{aligned}$$