



Induction

Lecture 21
CS2110 – Spring 2008

Overview

- Recursion
 - A programming strategy that solves a problem by reducing it to simpler or smaller instance(s) of the same problem
- Induction
 - A mathematical strategy for proving statements about natural numbers $0, 1, 2, \dots$ (or more generally, about inductively defined objects)
- They are very closely related
- Induction can be used to establish the *correctness* and *complexity* of programs

Defining Functions

- It is often useful to describe a function in different ways
 - Let $S : \text{int} \rightarrow \text{int}$ be the function where $S(n)$ is the sum of the integers from 0 to n . For example,

$$S(0) = 0 \qquad S(3) = 0+1+2+3 = 6$$

- Definition: iterative form

- ♦ $S(n) = 0+1+ \dots + n$

$$= \sum_{i=0}^n i$$

- Another characterization: closed form

- ♦ $S(n) = n(n+1)/2$

Sum of Squares

- A more complex example
 - Let $SQ : \text{int} \rightarrow \text{int}$ be the function that gives the sum of the squares of integers from 0 to n :

$$SQ(0) = 0 \quad SQ(3) = 0^2 + 1^2 + 2^2 + 3^2 = 14$$

- Definition (iterative form): $SQ(n) = 0^2 + 1^2 + \dots + n^2$
- Is there an equivalent closed-form expression?

Closed-Form Expression for $SQ(n)$

- Sum of integers between 0 through n was $n(n+1)/2$ which is a *quadratic* in n (that is, $O(n^2)$)
- Inspired guess: perhaps sum of *squares* of integers between 0 through n is a *cubic* in n
- Conjecture: $SQ(n) = an^3 + bn^2 + cn + d$ where a, b, c, d are unknown coefficients
- How can we find the values of the four unknowns?
 - Idea: Use any 4 values of n to generate 4 linear equations, and then solve



Finding Coefficients

$$SQ(n) = 0^2 + 1^2 + \dots + n^2 = an^3 + bn^2 + cn + d$$

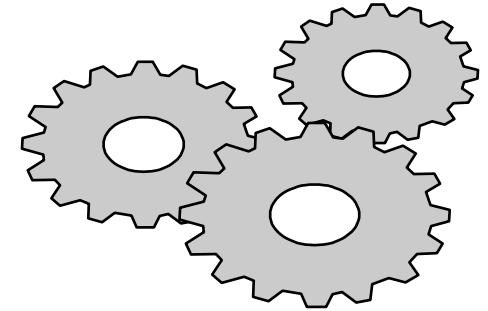
- Use $n = 0, 1, 2, 3$

$$SQ(0) = 0 = a \cdot 0 + b \cdot 0 + c \cdot 0 + d$$

$$SQ(1) = 1 = a \cdot 1 + b \cdot 1 + c \cdot 1 + d$$

$$SQ(2) = 5 = a \cdot 8 + b \cdot 4 + c \cdot 2 + d$$

$$SQ(3) = 14 = a \cdot 27 + b \cdot 9 + c \cdot 3 + d$$



Solve these 4 equations to get

$$a = 1/3 \quad b = 1/2 \quad c = 1/6 \quad d = 0$$

Is the Formula Correct?

- This suggests

$$\begin{aligned} \text{SQ}(n) &= 0^2 + 1^2 + \dots + n^2 \\ &= n^3/3 + n^2/2 + n/6 \\ &= n(n+1)(2n+1)/6 \end{aligned}$$

- Question: Is this closed-form solution true for all n ?
 - Remember, we only used $n = 0, 1, 2, 3$ to determine these coefficients
 - We do not know that the closed-form expression is valid for other values of n

One Approach

- Try a few other values of n to see if they work.
 - Try $n = 5$: $SQ(n) = 0+1+4+9+16+25 = 55$
 - Closed-form expression: $5 \cdot 6 \cdot 11 / 6 = 55$
 - Works!
- Try some more values...
- We can never prove validity of the closed-form solution for all values of n this way, since there are an infinite number of values of n

A Recursive Definition

- To solve this problem, let's express $SQ(n)$ in a different way:

$$SQ(n) = 0^2 + 1^2 + \dots + (n-1)^2 + n^2$$

- The part in the box is just $SQ(n-1)$

- This leads to the following *recursive* definition

$$SQ(0) = 0$$

$$SQ(n) = SQ(n-1) + n^2, \quad n > 0$$

- Thus,

$$\begin{aligned} SQ(4) &= SQ(3) + 4^2 = SQ(2) + 3^2 + 4^2 = SQ(1) + 2^2 + 3^2 + 4^2 = \\ &SQ(0) + 1^2 + 2^2 + 3^2 + 4^2 = 0 + 1^2 + 2^2 + 3^2 + 4^2 \end{aligned}$$

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Base Case

Recursive Case

- Thus,

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Are These Two Functions Equal?

- SQ_r (r = recursive)

$$SQ_r(0) = 0$$

$$SQ_r(n) = SQ_r(n-1) + n^2, \quad n > 0$$

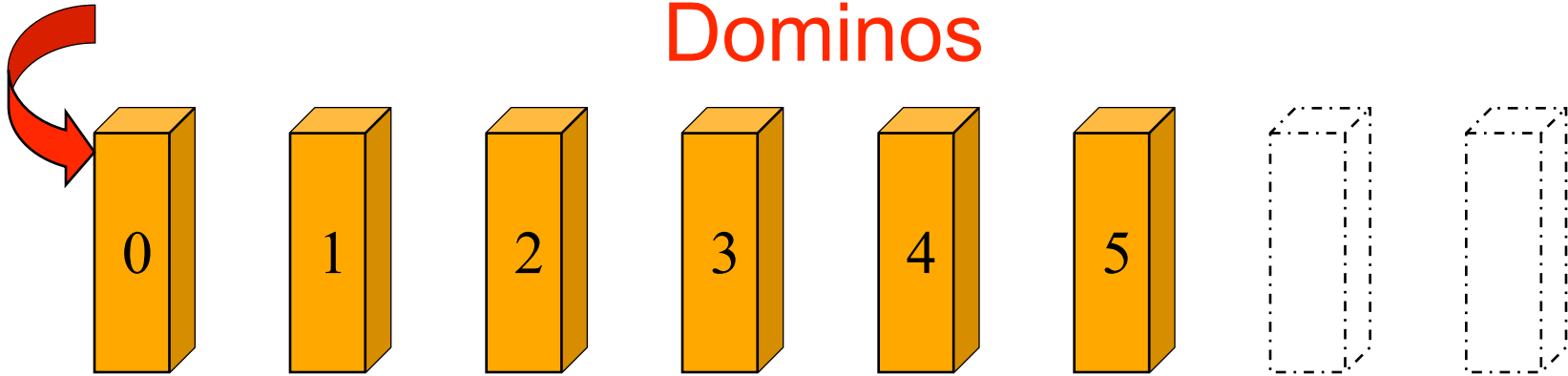
- SQ_c (c = closed-form)

$$SQ_c(n) = n(n+1)(2n+1)/6$$

Induction over Integers

- To prove that some property $P(n)$ holds for all integers $n \geq 0$,
 1. **Basis:** Show that $P(0)$ is true
 2. **Induction Step:** Assuming that $P(k)$ is true for an *unspecified* integer k , show that $P(k+1)$ is true
- **Conclusion:** Because we could have picked any k , we conclude that $P(n)$ holds for all integers $n \geq 0$

Dominos



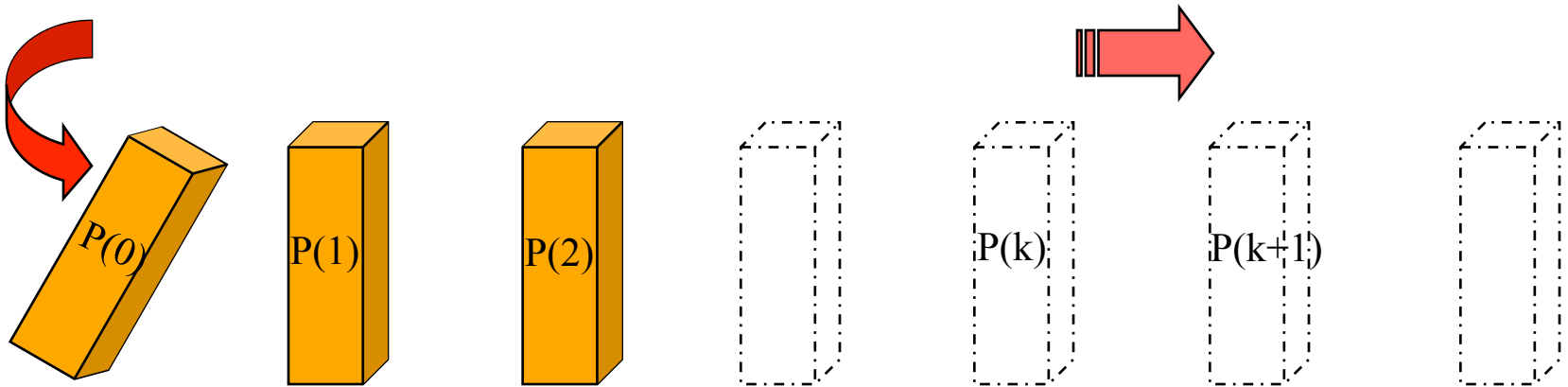
- Assume equally spaced dominos, and assume that spacing between dominos is less than domino length
- How would you argue that all dominos would fall?
- Dumb argument:
 - Domino 0 falls because we push it over
 - Domino 0 hits domino 1, therefore domino 1 falls
 - Domino 1 hits domino 2, therefore domino 2 falls
 - Domino 2 hits domino 3, therefore domino 3 falls
 - ...
- Is there a more compact argument we can make?

Better Argument

- Argument:
 - Domino 0 falls because we push it over (Base Case or Basis)
 - Assume that domino k falls over (Induction Hypothesis)
 - Because domino k 's length is larger than inter-domino spacing, it will knock over domino $k+1$ (Inductive Step)
 - Because we could have picked any domino to be the k^{th} one, we conclude that all dominoes will fall over (Conclusion)
- This is an inductive argument
- This version is called *weak induction*
 - There is also *strong induction* (later)
- Not only is this argument more compact, it works for an arbitrary number of dominoes!

$$SQ_r(n) = SQ_c(n) \text{ for all } n?$$

- Define $P(n)$ as $SQ_r(n) = SQ_c(n)$



- Prove $P(0)$
- Assume $P(k)$ for unspecified k , and then prove $P(k+1)$ under this assumption

Proof (by Induction)

Recall: $SQ_r(0) = 0$

$$SQ_r(n) = SQ_r(n-1) + n^2, \quad n > 0$$

$$SQ_c(n) = n(n+1)(2n+1)/6$$

Let $P(n)$ be the proposition that $SQ_r(n) = SQ_c(n)$

- **Basis:** $P(0)$ holds because $SQ_r(0) = 0$ and $SQ_c(0) = 0$ by definition
- **Induction Hypothesis:** Assume $SQ_r(k) = SQ_c(k)$

- **Inductive Step:**

$$\begin{aligned} SQ_r(k+1) &= SQ_r(k) + (k+1)^2 && \text{by definition of } SQ_r(k+1) \\ &= SQ_c(k) + (k+1)^2 && \text{by the Induction Hypothesis} \\ &= k(k+1)(2k+1)/6 + (k+1)^2 && \text{by definition of } SQ_c(k) \\ &= (k+1)(k+2)(2k+3)/6 && \text{algebra} \\ &= SQ_c(k+1) && \text{by definition of } SQ_c(k+1) \end{aligned}$$

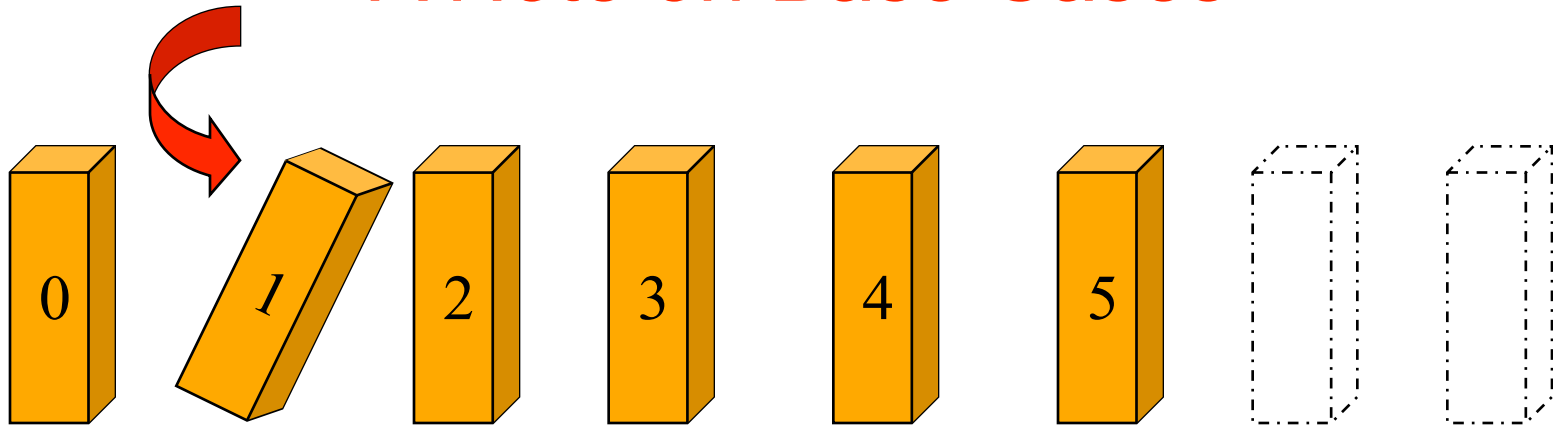
- **Conclusion:** $SQ_r(n) = SQ_c(n)$ for all $n \geq 0$

Another Example

Prove that $0+1+\dots+n = n(n+1)/2$

- **Basis:** Obviously holds for $n = 0$
- **Induction Hypothesis:** Assume $0+1+\dots+k = k(k+1)/2$
- **Inductive Step:**
$$\begin{aligned} 0+1+\dots+(k+1) &= [0+1+\dots+k] + (k+1) && \text{by def} \\ &= k(k+1)/2 + (k+1) && \text{by I.H.} \\ &= (k+1)(k+2)/2 && \text{algebra} \end{aligned}$$
- **Conclusion:** $0+1+\dots+n = n(n+1)/2$ for all $n \geq 0$

A Note on Base Cases



- Sometimes we are interested in showing some proposition is true for integers $\geq b$
- Intuition: we knock over domino b , and dominoes in front get knocked over; not interested in $0, 1, \dots, (b-1)$
- In general, the base case in induction does not have to be 0
- If base case is some integer b
 - Induction proves the proposition for $n = b, b+1, b+2, \dots$
 - Does not say anything about $n = 0, 1, \dots, b-1$

Weak Induction: Nonzero Base Case

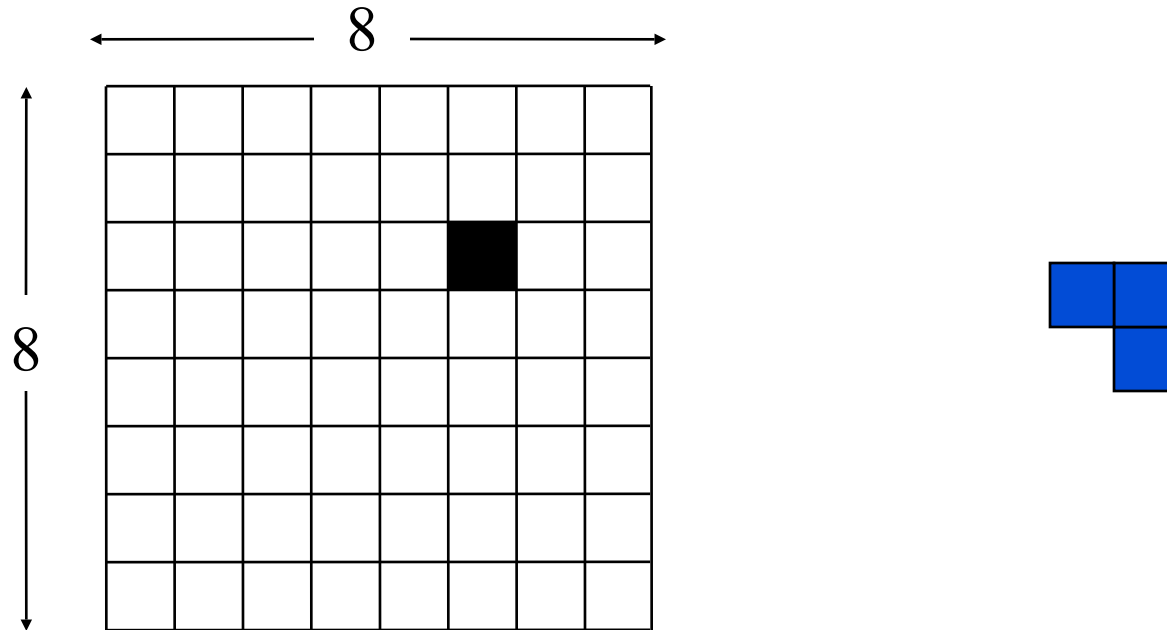
Claim: You can make any amount of postage above 8¢ with some combination of 3¢ and 5¢ stamps

- **Basis:** True for 8¢: $8 = 3 + 5$
- **Induction Hypothesis:** Suppose true for some $k \geq 8$
- **Inductive Step:**
 - If used a 5¢ stamp to make k , replace it by two 3¢ stamps. Get $k+1$.
 - If did not use a 5¢ stamp to make k , must have used at least three 3¢ stamps. Replace three 3¢ stamps by two 5¢ stamps. Get $k+1$.
- **Conclusion:** Any amount of postage above 8¢ can be made with some combination of 3¢ and 5¢ stamps

What are the “Dominos”?

- In some problems, it can be tricky to determine how to set up the induction
- This is particularly true for geometric problems that can be attacked using induction

A Tiling Problem



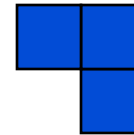
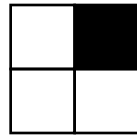
- A chessboard has one square cut out of it
- Can the remaining board be tiled using tiles of the shape shown in the picture (rotation allowed)?
- Not obvious that we can use induction!

Proof Outline

Consider boards of size $2^n \times 2^n$ for $n = 1, 2, \dots$

- **Basis:** Show that tiling is possible for 2×2 board
- **Induction Hypothesis:** Assume the $2^k \times 2^k$ board can be tiled
- **Inductive Step:** Using I.H. show that the $2^{k+1} \times 2^{k+1}$ board can be tiled
- **Conclusion:** Any $2^n \times 2^n$ board can be tiled, $n = 1, 2, \dots$
 - Our chessboard (8×8) is a special case of this argument
 - We will have proven the 8×8 special case by solving a more general problem!

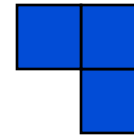
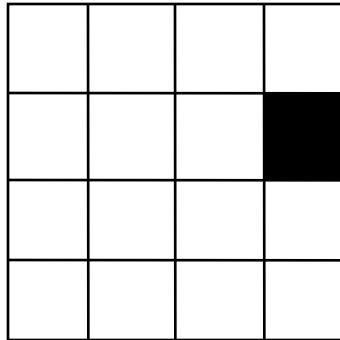
Basis



2 x 2 board

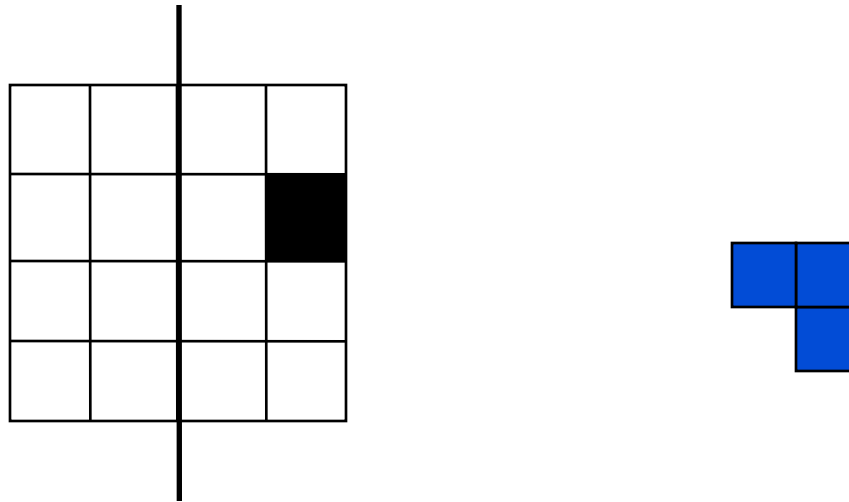
- The 2 x 2 board can be tiled regardless of which one of the four pieces has been omitted

4 x 4 Case



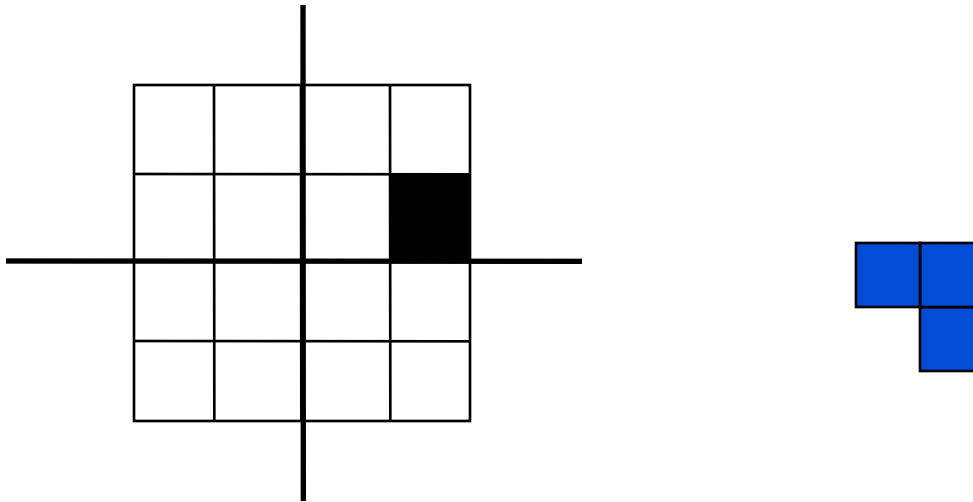
- Divide the 4 x 4 board into four 2 x 2 sub-boards
- One of the four sub-boards has the missing piece
 - By the I.H., that sub-board can be tiled since it is a 2 x 2 board with a missing piece
- Tile the center squares of the three remaining sub-boards as shown
 - This leaves three 2 x 2 boards, each with a missing piece
 - We know these can be tiled by the Induction Hypothesis

4 x 4 Case



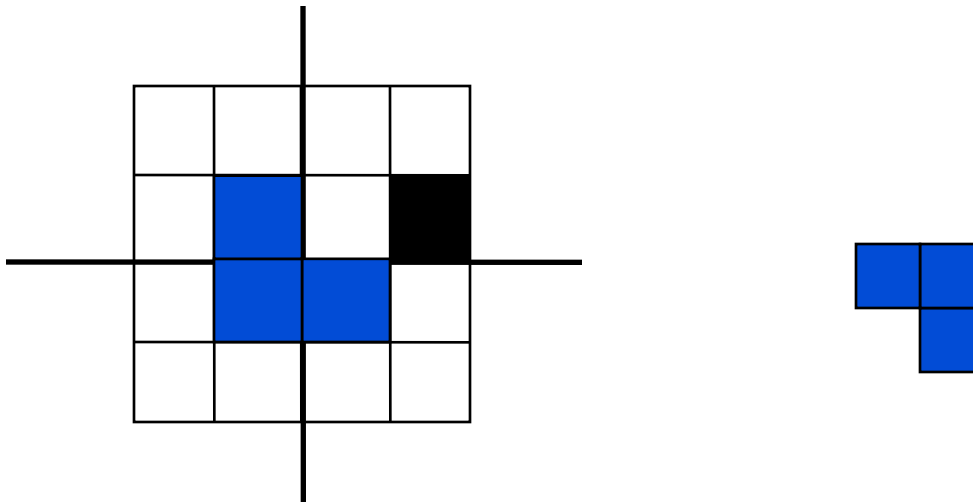
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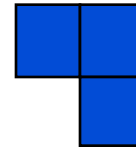
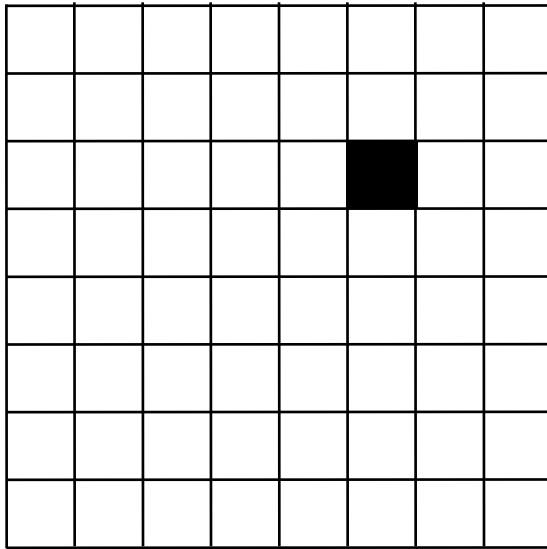
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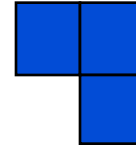
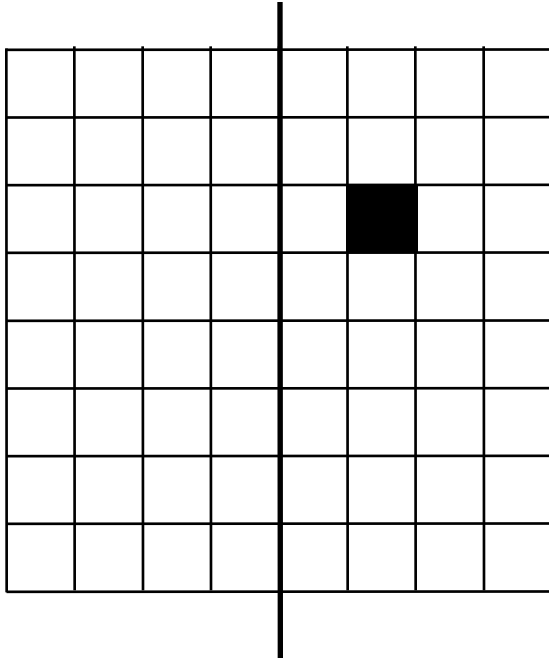
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$2^{k+1} \times 2^{k+1}$ case



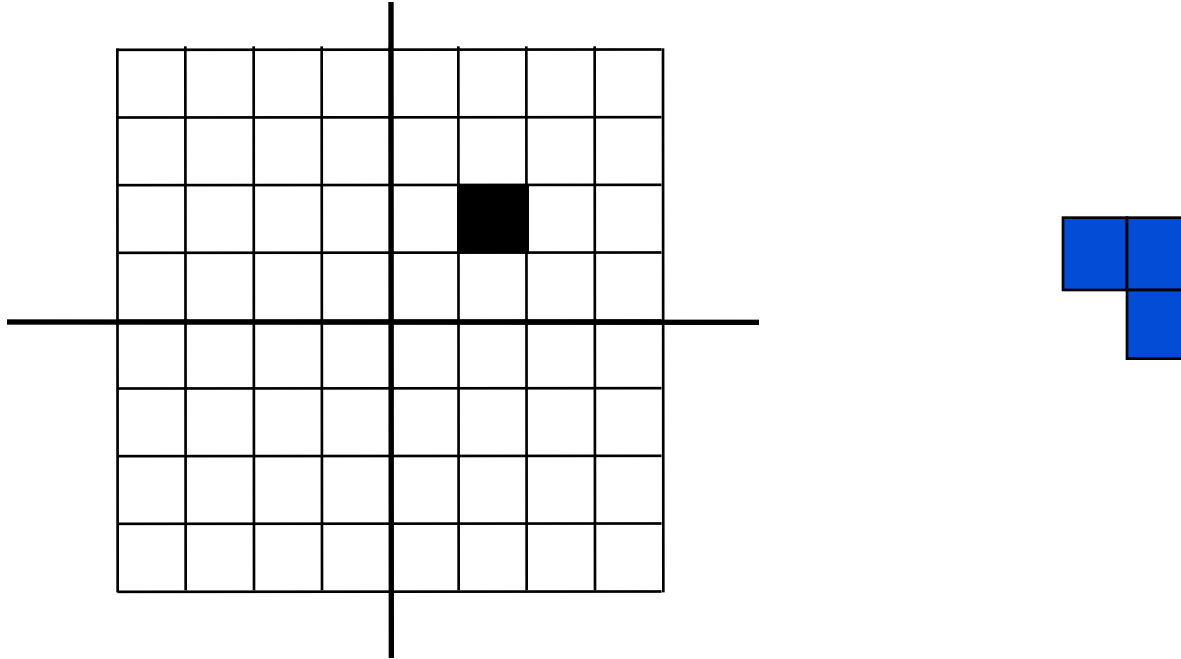
- Divide board into four sub-boards and tile the center squares of the three complete sub-boards
- The remaining portions of the sub-boards can be tiled by the I.H. (which assumes we can tile $2^k \times 2^k$ boards)

$2^{k+1} \times 2^{k+1}$ case



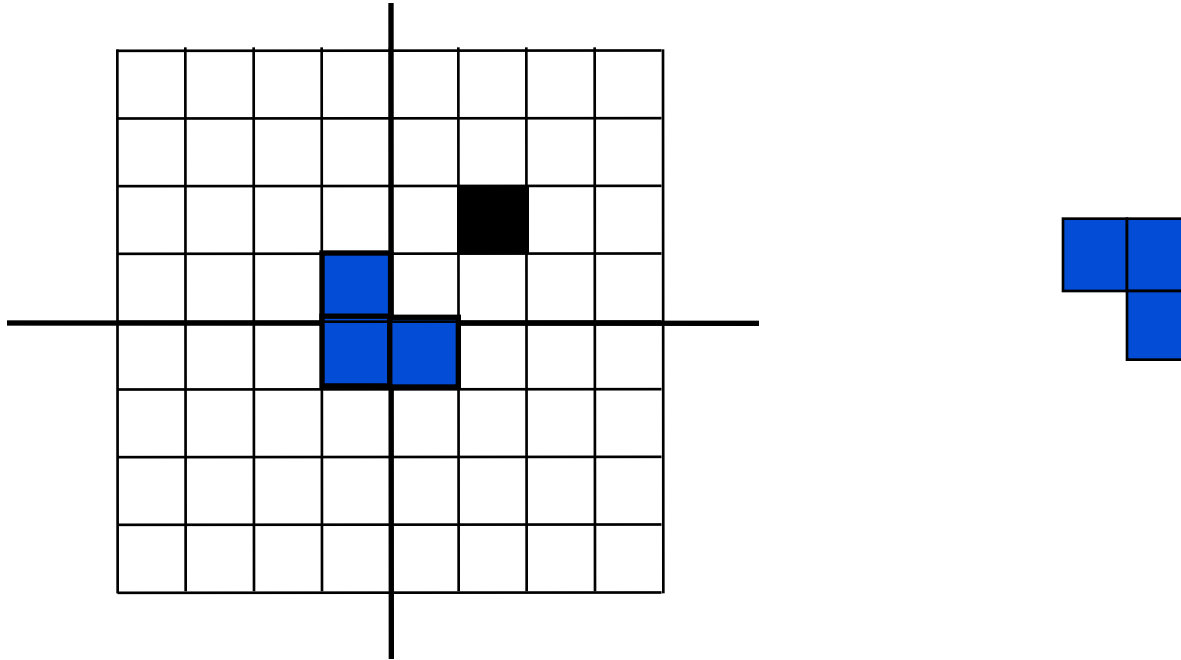
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When Induction Fails

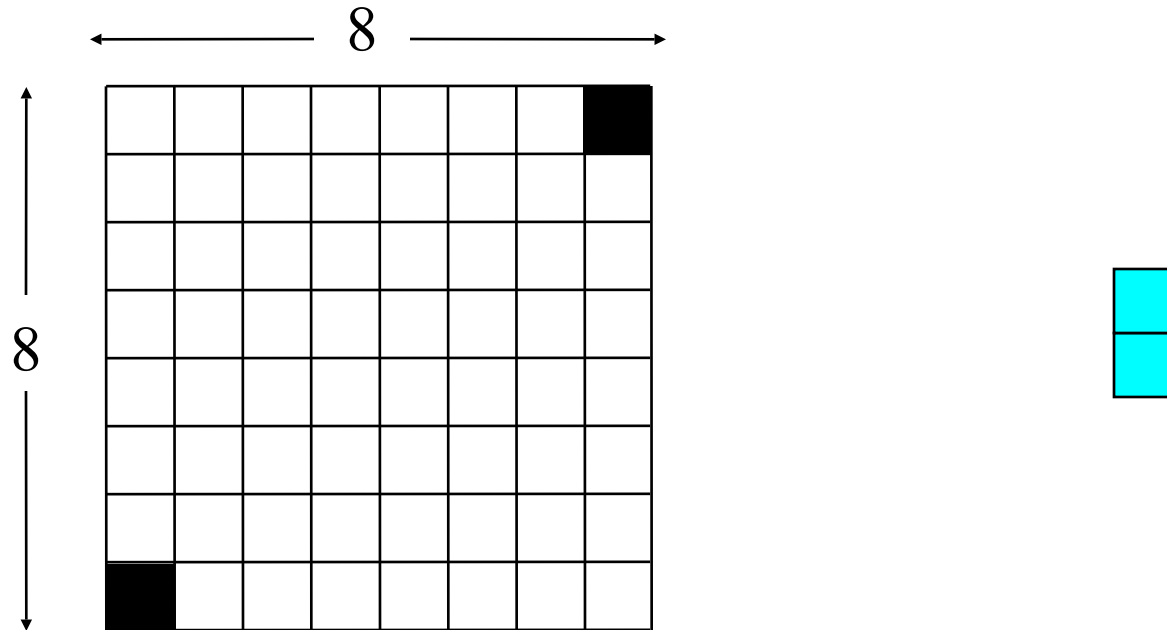
- Sometimes an inductive proof strategy for some proposition may fail
- This does not necessarily mean that the proposition is wrong
 - It may just mean that the particular inductive strategy you are using is the wrong choice
- A different induction hypothesis (or a different proof strategy altogether) may succeed

Tiling Example (Poor Strategy)

Let's try a different induction strategy

- Proposition
 - Any $n \times n$ board with one missing square can be tiled
- Problem
 - A 3×3 board with one missing square has 8 remaining squares, but our tile has 3 squares; tiling is impossible
- Thus, any attempt to give an inductive proof of this proposition *must fail*
- Note that this failed proof does not tell us anything about the 8×8 case

A Seemingly Similar Tiling Problem



- A chessboard has opposite corners cut out of it. Can the remaining board be tiled using tiles of the shape shown in the picture (rotation allowed)?
- Induction fails here. Why? (Well...for one thing, this board can't be tiled with dominos.)

Strong Induction

- We want to prove that some property P holds for all n
- Weak induction
 - $P(0)$: Show that property P is true for 0
 - $P(k) \Rightarrow P(k+1)$: Show that if property P is true for k , it is true for $k+1$
 - Conclude that $P(n)$ holds for all n
- Strong induction
 - $P(0)$: Show that property P is true for 0
 - $P(0)$ and $P(1)$ and ... and $P(k) \Rightarrow P(k+1)$: show that if P is true for numbers less than or equal to k , it is true for $k+1$
 - Conclude that $P(n)$ holds for all n
- Both proof techniques are equally powerful

Conclusion

- Induction is a powerful proof technique
- Recursion is a powerful programming technique
- Induction and recursion are closely related
 - We can use induction to prove correctness and complexity results about recursive programs