Recursion

Lecture 6
CS2110 – Fall 2008

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CS2110 F08 Quiz 1 -- 9/16/08

Name_____________________________NetID_______________

1. Suppose s1 and s2 are strings.  Choose the correct method of comparing them to
determine if they represent the same sequence of characters.

    A.  s1.equals(s2)         B.  s1 == s2         C.  s1 = s2

2. Fill in the truth table for the Boolean implication operator ->.

    P     Q     P->Q
   ==================
    T     T
    T     F
    F     T
    F     F

3. Which two of the following features of expression languages can be used to
avoid excessive parentheses?

    A. arity of operators         B. associativity of operators
    C. precedence of operators    D. equivalence of operators
    E. recursive-descent parsing

4. Write the following two Java fragments in one line:

    if (x.isEmpty()) {                    if (x == null) {
       return false;                         a = "no element available";
    } else {                              } else {
       return true;                          a = x.element;
    }                                     }
```

```
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    if (x.isEmpty()) {                    if (x == null) {
       return false;                         a = "no element available";
    } else {                              } else {
       return true;                          a = x.element;
    }                                     }
   return !x.isEmpty();    a = (x == null)? "no element available" : x.element;
```

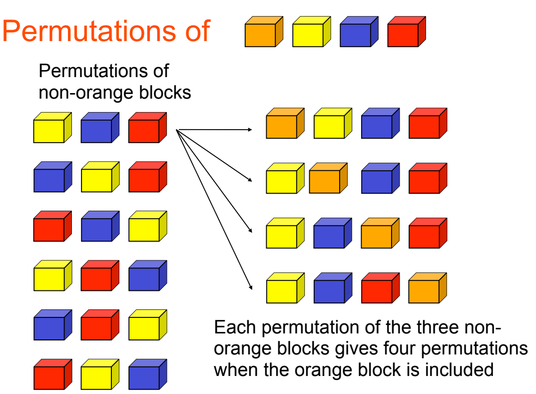
Recursion Overview

- Recursion is a powerful technique for specifying functions, sets, and programs
- Example recursively-defined functions and programs
  - factorial
  - combinations
  - exponentiation (raising to an integer power)
- Example recursively-defined sets
  - grammars
  - expressions
  - data structures (lists, trees, ...)

The Factorial Function (n!)

- Define $n! = n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1$ read: "n factorial"
  - E.g., $3! = 3\cdot 2\cdot 1 = 6$
- By convention, $0! = 1$
- The function int → int that gives n! on input n is called the factorial function
- n! is the number of permutations of n distinct objects
  - There is just one permutation of one object. 1! = 1
  - There are two permutations of two objects: 2! = 2
    1 2   2 1
  - There are six permutations of three objects: 3! = 6
    1 2 3   1 3 2   2 1 3   2 3 1   3 1 2   3 2 1
- If n > 0, $n! = n\cdot(n-1)!$

Permutations of



Total number = 4·6 = 24 = 4!

A Recursive Program

0! = 1

n! = n·(n–1)!, n > 0

```
static int fact(int n) {
   if (n == 0) return 1;
   else return n*fact(n-1);
}
```

Execution of fact(4)

fact(4) → 24
fact(3) → 6
fact(2) → 2
fact(1) → 1
fact(0) → 1

General Approach to Writing Recursive Functions

1. Try to find a parameter, say n, such that the solution for n can be obtained by combining solutions to the *same problem using smaller values of n* (e.g., (n-1)!)
2. Find *base case(s)* – small values of n for which you can just write down the solution (e.g., 0! = 1)
3. Verify that, for any valid value of n, applying the reduction of step 1 repeatedly will ultimately hit one of the base cases

The Fibonacci Function

- Mathematical definition:
  - fib(0) = 0
  - fib(1) = 1
  - fib(n) = fib(n – 1) + fib(n – 2), n ≥ 2

  (fib(0), fib(1): two base cases!)
- Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, …

```
static int fib(int n) {
   if (n == 0) return 0;
   else if (n == 1) return 1;
   else return fib(n-1) + fib(n-2);
}
```

Fibonacci (Leonardo Pisano) 1170–1240?

Statue in Pisa, Italy
Giovanni Paganucci
1863

Recursive Execution

```
static int fib(int n) {
   if (n == 0) return 0;
   else if (n == 1) return 1;
   else return fib(n-1) + fib(n-2);
}
```

Execution of fib(4):

fib(4)
- fib(3)
  - fib(2)
    - fib(1)
    - fib(0)
  - fib(1)
- fib(2)
  - fib(1)
  - fib(0)

Combinations (a.k.a. Binomial Coefficients)

- How many ways can you choose r items from a set of n distinct elements? $\binom{n}{r}$ "n choose r"

  $\binom{5}{2}$ = number of 2-element subsets of {A,B,C,D,E}

  2-element subsets containing A: $\binom{4}{1}$
  {A,B}, {A,C}, {A,D}, {A,E}

  2-element subsets not containing A: {B,C},{B,D},{B,E},{C,D},{C,E},{D,E} $\binom{4}{2}$

- Therefore, $\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$

Combinations

$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$, n > r > 0

$\binom{n}{n} = 1$

$\binom{n}{0} = 1$

Can also show that $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Pascal's triangle =

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Binomial Coefficients

- Combinations are also called *binomial coefficients* because they appear as coefficients in the expansion of the binomial power $(x+y)^n$:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i$$

Combinations Have Two Base Cases

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$

$$\binom{n}{n} = 1$$

$$\binom{n}{0} = 1$$

Two base cases

- Coming up with right base cases can be tricky!
- General idea:
  - Determine argument values for which recursive case does not apply
  - Introduce a base case for each one of these

Recursive Program for Combinations

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$

$$\binom{n}{n} = 1$$

$$\binom{n}{0} = 1$$

```
static int combs(int n, int r) {    //assume n>=r>=0
    if (r == 0 || r == n) return 1; //base cases
    else return combs(n-1,r) + combs(n-1,r-1);
}
```


Positive Integer Powers

- $a^n = a \cdot a \cdot a \cdots a$ (n times)
- Alternate description:
  - $a^0 = 1$
  - $a^{n+1} = a \cdot a^n$

```
static int power(int a, int n) {
    if (n == 0) return 1;
    else return a*power(a,n-1);
}
```


A Smarter Version

- Power computation:
  - $a^0 = 1$
  - If n is nonzero and even, $a^n = (a^{n/2})^2$
  - If n is odd, $a^n = a \cdot (a^{n/2})^2$
    - Java note: If x and y are integers, "x/y" returns the integer part of the quotient
- Example:

  $a^5 = a \cdot (a^{5/2})^2 = a \cdot (a^2)^2 = a \cdot ((a^{2/2})^2)^2 = a \cdot (a^2)^2$

  Note: this requires 3 multiplications rather than 5!

- What if n were larger?
  - Savings would be more significant
- This is much faster than the straightforward computation
  - Straightforward computation: n multiplications
  - Smarter computation: log(n) multiplications

Smarter Version in Java

- n = 0: $a^0 = 1$
- n nonzero and even: $a^n = (a^{n/2})^2$
- n nonzero and odd: $a^n = a \cdot (a^{n/2})^2$

parameters

local variable

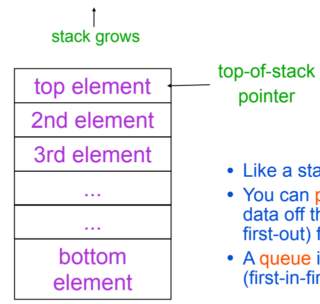
```
static int power(int a, int n) {
    if (n == 0) return 1;
    int halfPower = power(a,n/2);
    if (n%2 == 0) return halfPower*halfPower;
    return halfPower*halfPower*a;
}
```

- The method has two parameters and a local variable
- Why aren't these overwritten on recursive calls?

Implementation of Recursive Methods

- Key idea:
  - Use a stack to remember parameters and local variables across recursive calls
  - Each method invocation gets its own stack frame
- A stack frame contains storage for
  - Local variables of method
  - Parameters of method
  - Return info (return address and return value)
  - Perhaps other bookkeeping info

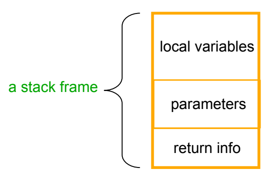
Stacks



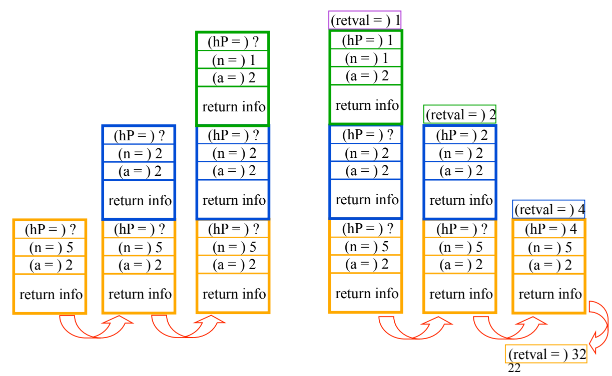
- Like a stack of plates
- You can push data on top or pop data off the top in a LIFO (last-in-first-out) fashion
- A queue is similar, except it is FIFO (first-in-first-out)

Stack Frame

- A new stack frame is pushed with each recursive call
- The stack frame is popped when the method returns
  - Leaving a return value (if there is one) on top of the stack



Example: power(2, 5)

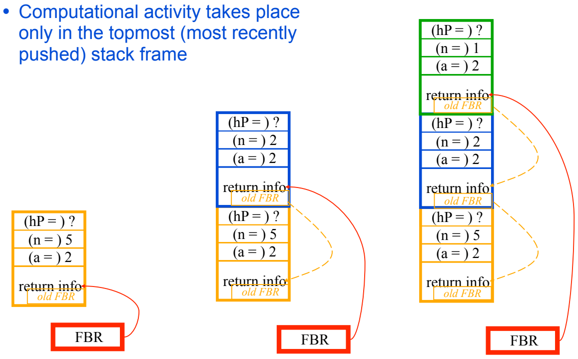


How Do We Keep Track?

- At any point in execution, many invocations of *power* may be in existence
  - Many stack frames (all for *power*) may be in Stack
  - Thus there may be several different versions of the variables *a* and *n*
- How does processor know which location is relevant at a given point in the computation?
- Answer: Frame Base Register
  - When a method is invoked, a frame is created for that method invocation, and FBR is set to point to that frame
  - When the invocation returns, FBR is restored to what it was before the invocation
- How does machine know what value to restore in the FBR?
  - This is part of the return info in the stack frame

FBR

- Computational activity takes place only in the topmost (most recently pushed) stack frame



Conclusion

- Recursion is a convenient and powerful way to define functions
- Problems that seem insurmountable can often be solved in a “divide-and-conquer” fashion:
  - Reduce a big problem to smaller problems of the same kind, solve the smaller problems
  - Recombine the solutions to smaller problems to form solution for big problem
- Important application (next lecture): parsing