# BSTs and Balanced Trees

CS211
Fall 2000

---

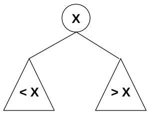
# Quadratic Probing + Hashing Pitfalls

- Quadratic Probing
  - Similar to Linear Probing in that data is stored within the table
  - Probe at h(X), then at
    - h(X)+1
    - h(X)+4
    - h(X)+9
    - …
    - $h(X)+ i^2$
  - Works well when
    - ▲ $\lambda < 0.5$
    - ▲ table size is *prime*
- Hash Table Pitfalls
  - Good hash function is *required*
  - Watch the load factor ($\lambda$), especially for Linear & Quadratic Probing

---

# Dictionary Implementations

- Ordered Array
  - Better than unordered array because Binary Search can be used
- Unordered Linked-List
  - Ordering doesn't help
- Direct Address Table
  - Small universe ⇒ limited usage
- Hashtables
  - O(1) expected time for Dictionary operations
- Goal: Want Binary Search, but can't afford inefficiency of ordered array
- Idea: Use a Binary Search Tree (BST)
- BST Property:



---

# Deleting from a BST

Cases:

- Delete a leaf
  - easy
- Delete a node with just one child
  - delete and replace with child
- Delete a node with two children
  - delete node's successor
  - write successor's data into node
- How do we find the successor?
- The successor always has at most one child. Why?
- Would work just as well using predecessor instead of successor

---

# BST Performance

- Time for put(), get(), update(), remove() is O(*h*) where *h* is the height of the tree
- How bad can *h* be?
- Operations are fast if tree is *balanced*
- How balanced is a random tree?
  - If items are inserted in random order then the expected height of a BST is O(log *n*) where *n* is the number of items
- If deletion is allowed
  - Tree is no longer random
  - Tree is likely to become unbalanced

---

# Analysis Sketch for Random BST

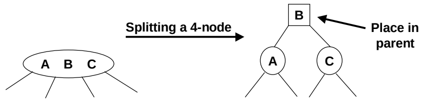
- Only the number of items and their order is important
  - Can restrict our attention to BSTs containing items {1,…, *n*}
- We assume that each item is equally likely to appear as the root
- Define H(*n*) ≡ expected height of BST of size *n*
- If item *i* is the root then expected height is
  $1 + \max \{ H(i-1), H(n-i) \}$
  We average this over all possible *i*
- Can solve the resulting recurrence (by induction)
  $H(n) = O(\log n)$

## Why use a BST instead of a Hashtable?

- If we use a *balanced* BST scheme then we achieve guaranteed *worst-case* time bounds
- There are some operations that can be efficient on BSTs, but very inefficient on Hashtables
  - report-elements-in-order
  - getMin
  - getMax
  - select(k) // find the *k*-th element
    - (maintain size of each subtree by using an additional *size* field in each node)
- Note that balanced BST schemes can be difficult to implement
  - But there are lots of reliable codes for these schemes available on the Web
  - Java 1.2 includes a balanced BST scheme among its standard packages (java.util.TreeMap and java.util.TreeSet)

## Example Balancing Scheme: 234-Trees

- Nodes have 2, 3, or 4 children (and contain 1, 2, or 3 keys, respectively)
- All leaves are at the same level
- Basic rule for insertion: We hate 4-nodes
  - Split a 4-node whenever you find one while coming down the tree
  - Note: this requires that parent is not a 4-node
- Delete is harder than insert
  - For delete, we hate 2-nodes
  - As in BSTs, cannot delete from a nonleaf so we use same BST trick: delete successor and recopy its data



## 234-Tree Analysis

- Time for insert or get is proportional to tree's height
- How big is tree's height *h*?
- Let *n* be the number of nodes in the tree
  - n is large if all nodes are 4-nodes
  - n is small if all nodes are 2-nodes
- Can use this to show h = O(log n)

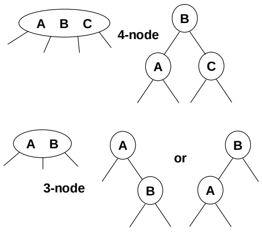
Analysis of tree height:

- Let *N* be the number of nodes, *n* be the number of items, and *h* be the height
- Define *h* so that a tree consisting of a single node is height 0
- It's easy to see $1+2+4+\ldots+2^h \leq N \leq 1+4+16+\ldots+4^h$
- It's also easy to see $N \leq n \leq 3N$
- Using the above, we have $n \geq 1+2+4+\ldots+2^h = 2^{h+1}-1$
- Rewriting, we have $h \leq \log(n+1) - 1$ or $h = O(\log n)$
- Thus, Dictionary operations on 234-trees take time O(log n) in the worst case

## 234-Tree Implementation

- Can implement all nodes as 4-nodes
  - Wasted space
- Can allow various node sizes
  - Requires recopying of data whenever a node changes size
- Can use BST nodes to emulate 2-, 3-, or 4-nodes

## Using BSTs to Emulate 234-Trees



- A 2-node can be represented with a standard BST node
- A 4-node can be represented with three BST nodes
- A 3-node can be represented with two BST nodes (in two different ways)

## Red-Black Trees

- We need a way to tell when an emulated 234-node starts and ends
- We mark the nodes
  - Black: "root" of 234-node
  - Red: belongs to parent
  - Requires one bit per node
- 234-tree rules become rules for rotations and color changes in red-black trees
- Result:
  - one black node per 234-node
  - Number of black nodes on path from root to leaf is same as height of 234-tree
  - All paths from root to leaf have same number of black nodes
  - On any path: at most one red node per black node
  - Thus tree height for red-black tree is O(log n)

## Balanced Tree Schemes

- AVL trees [1962]
  - named for initials of Russian creators
  - uses rotations to ensure heights of child trees differ by at most 1
- 23-Trees [Hopcroft 1970]
  - similar to 234-tree, but repairs have to move back up the tree
- B-Trees [Bayer & McCreight 1972]
- Red-Black Trees [Bayer 1972]
  - not the original name
- Red-black convention & relation to 234-trees [Guibas & Stolfi 1978]
- Splay Trees [Sleator & Tarjan 1983]
- Skip Lists [Pugh 1990]
  - developed at Cornell

## Selecting a Dictionary Scheme

- Use an unordered array for small sets (< 20 or so)
- Use a Hash Table if possible
  - Cannot efficiently do some ops that are easy with BSTs
  - Running times are expected rather than worst-case
- Use an ordered array if few changes after initialization
- B-Trees are best for large data sets, external storage
  - Widely used within data base software
- Otherwise, Red-Black Trees are current scheme of choice
- Skip Lists are supposed to be easier to implement
  - But shouldn't have to implement—use existing code
- Splay trees are useful if some items are accessed more often than others
  - But if you know which items are most-commonly accessed, use a separate data structure

## Selecting a Priority Queue Scheme

- Use an unordered array for small sets (< 20 or so)
- Use a sorted array or sorted linked list if few insertions are expected
- Use an array of linked lists if there are few priorities
  - Each linked list is a queue of equal-priority items
  - Very easy to implement
- Otherwise, use a Heap if you can
- Heap + Hashtable
  - Allow *change-priority* operation to be done in O(log n) expected time
- Balanced tree schemes
  - Useful and practical
- There are a number of alternate implementations that allow additional operations
  - Fibonacci heaps
  - Skew heaps
  - …

## Topics Covered Since Last Exam

- Data structure building blocks
  - Arrays, Lists, Trees, Graphs
- The Java Collections Framework
  - Interfaces: Set, SortedSet, List, Map, SortedMap, Iterator
  - Classes: HashSet, TreeSet, ArrayList, LinkedList, HashMap, TreeMap
  - Utilities: java.util.Arrays, java.util.Collections
- GUIs
  - Layout
  - Event handling
- Priority Queues
  - Heaps
  - BSTs, Balanced Trees
  - Array of lists (index value is priority)
- Dictionaries
  - BSTs
  - Balanced Trees
  - Hash Tables