Algorithmic Game Theory	**Problem Set 3**
CS 6840 Spring 2014	**Due Friday, March 21st**

The questions on this problem set are of varying difficulty. For full credit, you need to solve at least 4 of the 5 problems below. A full solution for each problem includes proving that your answer is correct. If you cannot solve a problem, write down how far you got, and why are you stuck.

You may discuss the questions with other students, but need to write down the solution yourself. Please acknowledge the students you discussed the questions with on your write-up. You may use any fact we proved in class without proving the proof or reference, and may read the relevant chapters of the book. However, you may **not use other published papers, or the Web to find your answer**.

Solutions can be submitted on CMS in pdf format (only). Please type your solution or write extremely neatly to make it easy to read. If your solution is complex, say more than about half a page, please include a 3-line summary to help us understand the argument.

We will post answers to questions on Piazza.

(1) Consider an extension of the cost-sharing problem from March 5th. Suppose the game has n players. In class we had each congestible element $e \in E$ have a fixed cost c_e, and the cost of the edge when $k \leq n$ users share the edge was $c_e(k) = c_e/k$. Now assume that the cost of providing the service is not fixed c_e, but rather grows with the number of users. Let $totalC_e(k)$ denote the total cost with k users. We will assume that $totalC_e(0) = 0$ and $totalC_e(k)$ is monotone non-decreasing with k. As before, we will use fair-sharing to share the cost, and charge each of the k users $c_e(k) = totalC_e(k)/k$, when k users share the edge. Notice that this remains a congestion game for any function $totalC_e(k)$

(a) Suppose the functions $totalC_e(k)$ are all concave, implying that $totalC_e(k+1) - totalC_e(k)$ is monotone decreasing (non-increasing) in k. Show that the price of stability bound of $H(n) = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$, we shown in class for the special case when the cost is fixed, also extends to this more general class of problems.

(b) Does the price of stability bound claimed in part (a) also apply without the concave assumption on the cost? How bad can the price of stability get without this assumption?

(2) Consider a routing game (congestion game) with linear congestion costs $c_e(x) = a_e x + b_e$ on each edge e for nonnegative integers a_e and b_e. Assume that there are k players, and each player wants to route a unit of flow on a single path (atomic congestion game). Recall that this game is $(5/3, 1/3)$-smooth, and hence has a price of anarchy bounded by at most $\frac{5}{2}$.

(a) Show that the price of stability of this game is at most 2, i.e., that the game has a Nash equilibrium that has cost at most twice the minimum possible cost. Hint: take advantage of the potential function.

(b) Next, consider the game play when at each iteration, a player is selected and the selected player best responds to the current play by selecting the path that is currently cheapest.

Show that if the total costs $c(s) = \sum_i c_i(s)$ is more than 2.5 times the minimum possible cost at a strategy vector s, then some player i has an alternate strategy s'_i such that

$$c_i(s) - c_i(s'_i, s_{-i}) \geq \frac{1}{3k}(2c(s) - 5c(s^*)).$$

(c) Suppose at each iteration, the player that can make the maximum possible improvement in his cost will best respond. Let C be a bound on the maximum cost of any path, say $C = n \max_e a_e k + b_e$, where n is the number of nodes in the graph. Show that after $O(k \log(\epsilon^{-1} C))$ steps the total cost of the solution will be bounded by at most $(2.5 + \epsilon)c(s^*)$. Hint: use the connection between cost and the potential function from part (a).

(3) An interesting question in the cost-sharing game from March 5th is the possible positive effect of cooperation. For this problem, consider only the original version with fixed cost c_e. In the bad example of two parallel edges that all players have to choose between, cooperation would lead to the optimal solution. To be concrete, we define strong Nash equilibrium when not only single players do not want to deviate, but no group of players have incentive to deviate together. More precisely, we will consider a cost minimization game. For two strategy vectors s and s' and a set of players A we will consider the vector (s'_A, s_{-A}) where players in set A use strategy s' while players outside A use the old strategy s. We think of this as the outcome when all players in A collectively deviate to strategy s'. We say that a strategy vector s is a *strong Nash equilibrium*, if for any subset of players A, and any alternate strategy vector s' there exists a player $i \in A$ such that $c_i(s'_A, s_{-A}) > c_i(s)$, i.e, for any group of players A, and any possible deviation s', some player in A rather stay with the original strategy.

(a) Consider the game of the Prisoner Dilemma discussed in the first class. Recall that this game has a unique Nash equilibrium when both players confess. Is this Nash equilibrium a strong Nash? Does this game have a strong Nash? please explain your answer.

(b) Consider a cost-sharing game with fixed cost c_e for each element $e \in E$. Let s be a strong Nash equilibrium, and s^* be an optimal solution. Prove that there is a player i such that $c_i(s) \leq c_i(s^*)$, i.e., whose cost in the strong Nash is less than or equal to his cost in the optimal solution.

(c) Show that there is an ordering of the players such that if we use OPT_i to denote the optimal solution of just the first i players (when players $j > i$ do not participate in the game), and use $c_j(OPT_i)$ to denote the cost of player j in this solution (for $j \leq i$), then $c_i(s) \leq c_i(OPT_i)$ for all i. Note that part (b) is the special case of this for $i = n$.

(d) Use parts (b-c) to show that the total cost of any strong Nash equilibrium in this game is at most $H(n)$ times the optimal cost. Hint: use the fact that cost-sharing is a potential game with potential function $\Phi(s)$ that satisfies $cost(s) \leq \Phi(s) \leq H(n)cost(s)$.

(4) Consider a problem of selling k identical items to n bidders. Assume this time that a bidder may be interested in more than one item. Suppose bidder i is interested in n_i copies of the item, and has value v_i for getting this many items. We assume bidders are "single minded" and have no value for fewer than n_i items. The VCG auction requires to find the subset I of maximum total value $\sum_{i \in I} v_i$, subject to the restriction that $\sum_{i \in I} n_i \leq k$. Here we use variants of a greedy method.

(a) For this part, we consider only the optimization aspect of the problem. Assume that v_i and n_i are known, and consider the greedy mechanism that sorts agents in decreasing order of v_i/n_i and assigns them items "till the supplies last". Assume that each n_i is small compared to k, say $cn_i \leq k$ for a constant $c > 1$, and show that this algorithm is a $1 - 1/c$ approximation algorithm, that is, finds a solution that has value at least a $\frac{c}{c-1}$ of the maximum possible.

(b) Now consider the greedy algorithm in the game theoretic context. For the remainder of this problem, assume that n_i is public knowledge, and only v_i is private. Consider the following mechanism. Ask players for a bid b_i (willingness to pay), and sort by b_i/n_i, and assigns goods to agents till the supplies last. The traditional definition of critical price would assign the minimal the per-unit price at which the bidder is still getting assigned his n_i unit. Is the greedy mechanism with this price truthful (that is, is bidding the true value always a Nash equilibrium)?

(c) A simpler version of "critical price" charges each agent the minimal per-unit price that he/she had to bid to keep his spot in the sorted order. More formally, if agent i requires n_i units, and is right before agent j in the sorted order, then the price i gets changed is $n_i \frac{b_j}{n_j}$. Is this mechanism truthful?

(d) Consider a Nash equilibrium of the above mechanism, where we assume $b_i \leq v_i$ for all i. (Note that bidding above v_i is dominated). Show a bound on the price of anarchy for this mechanism. That is, bound the total value $\sum v_i$ of all agents who are assigned their required items in Nash compared to the highest possible. You may want to start with the special case when $n_i = 1$ for all i, and see how this generalizes to agents requiring larger bundles.

(e) Does your analysis of part (d) extend to the case when the quantity n_i is also private, and agents need to announce (q_i, b_i) pairs as bids? Assume there is free disposal, so getting more than n_i items is just as useful as getting exactly n_i items, but agents have no value for less than their required n_i items.

(5) Give a lower bound on the Bayesian price of anarchy of the first-price auction (for independent but non-identical distributions). Extra credit: Give a lower bound that exceeds the best known lower bound of about 1.05.