Price of Anarchy in Utility Max.

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Remind smoothness for cost minimization games and PoA theorem.

- Utility Maximization: n players, strategy space Σ_i, utility: $\Sigma_1 \times \dots \times \Sigma_n \to R$
- PNE: A strategy profile $s \in \Sigma_1 \times \dots \times \Sigma_n$ s.t.

  $\forall i, \forall s_i'$: $u_i(s) \geq u_i(s_i', s_{-i})$
- Price of Anarchy: $\max_{s \text{ is Eq.}} \frac{SW(OPT)}{SW(s)}$ $\qquad SW(s) = \sum_i u_i(s)$

- <u>Example:</u> Market Sharing Game

n firms, m markets

Each market j has a total demand v_j

If k firms invest in the same market
demand is shared equally: $\frac{v_j}{k}$

- Strategy space $\Sigma_i = \{1, \dots, m\}$
- $u_i(s) = \frac{v_{s_i}}{n_{s_i}(s)}$ $\qquad n_j(s) = |\{i : s_i = j\}|$
- $SW(s) = \sum_{j \in s_1 \cup s_2 \cup \dots \cup s_n} v_j = V(s_1 \cup s_2 \cup \dots \cup s_n)$

<u>Note:</u> $V(S) = \sum_{t \in S} v_t$

- <u>Thm</u> Market Sharing is Potential Game

<u>Pf</u> Why??

It is a congestion game (in the utility version)

$$\phi(s) = \sum_{j \in M} \sum_{t=1}^{n_j(s)} \frac{v_j}{t}$$

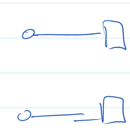
- <u>Cor</u> PNE always exists.

- <u>Instance:</u>

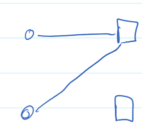
1 o $\qquad$ A $\square$ $v_1 = 2$

2 o $\qquad$ B $\square$ $v_2 = 1^-$

OPT: $SW = 3$



<u>Eq:</u> $SW = 2$.



- $u_1 = u_2 = 1 \geq u_1(B, A) = 1^-$

∘ $PoA = \frac{3}{2}$.

▷ $PoA \le ?$ Let's use the smoothness framework from last time.

▷ Smoothness for utility maximization games:

(λ, μ)-smooth if $\exists$ optimal s^* s.t. $\forall s$

$$\sum_i u_i(s_i^*, s_{-i}) \ge \lambda\, SW(s^*) - \mu\, SW(s)$$

$$(\text{contr}) \quad \sum_i c_i(s_i^*, s_{-i}) \le \lambda\, SC(s^*) + \mu\, SC(s)$$

▷ Thm) If utility-max game (λ, μ)-smooth then

$$PoA \le \frac{\lambda}{1+\mu}.$$

Pf) Let s be PNE:

$$u_i(s) \ge u_i(s_i^*, s_{-i})$$

$$SW(s) = \sum_i u_i(s) \ge \sum_i u_i(s_i^*, s_{-i})$$

$$\ge \lambda\, SW(s^*) - \mu\, SW(s)$$

$$\Rightarrow \quad SW(s) \ge \frac{\lambda}{1+\mu} SW(s^*)$$

$\blacksquare$

▷ Back to market sharing:

Thm) Market Sharing is $(1,1)$-smooth. $\Rightarrow PoA \le 2$.

Pf)

$$u_i(s_i^*, s_{-i}) \ge v_{s_i^*} \mathbb{1}_{\{s_i^* \notin S_{-i}\}} \qquad S_{-i} = \bigcup_{i' \ne i} s_{i'} \qquad \left(u_i(s_i^*, s_{-i}) \ge V(s_i^* \mid S_{-i}) \right)$$

$$\Rightarrow \sum_i u_i(s_i^*, s_{-i}) \ge \sum_i v_{s_i^*} \mathbb{1}_{\{s_i^* \notin S_{-i}\}} \qquad \left(\sum_i V(s_i^* \mid S_{-i}) \right)$$

$$\ge \sum_i v_{s_i^*} \mathbb{1}_{\{s_i^* \notin S\}} \qquad \left(\sum_i V(s_i^* \mid S) \right)$$

$$\boxed{S_{<i}^* = s_1^* \cup s_2^* \cup \ldots \cup s_{i-1}^*} \quad \ge \sum_i v_{s_i^*} \mathbb{1}_{\{s_i^* \notin S \cup S_{<i}^*\}} \qquad \left(\sum_i V(s_i^* \mid S \cup S_{<i}^*) \right)$$

$$= \sum V(S \cup S_{\le i}^*) - V(S \cup S_{\le i-1}^*)$$

$S \ldots s \cup \ldots \cup s \quad = V(S \cup S^*) - V(S)$

$\{ s_i^* \notin S \cup S_{<i} \}$

$$\boxed{S = s_1 \cup \dots \cup s_n}$$

$$= V(S \cup S^*) - V(S)$$
$$\geq V(S^*) - V(S)$$
$$= SW(s^*) - SW(s)$$

$$= \sum_i V(S \cup S^*_{\leq i}) - V(S \cup S^*_{\leq i-1})$$
$$= V(S \cup S^*) - V(S)$$

$\blacksquare$

▷ General Utility framework:

Spse $\quad SW(s) = V(S)$

Also: $\quad u_i(s_i, s_{-i}) \geq V(S) - V(S_{-i}) = V(s_i \mid S_{-i})$ (marg. contrib)

Also: $\quad V(S)$ is submodular and monotone

Thm | Then game is $(1,1)$-smooth $\Rightarrow PoA \leq 2$

▷ Tightness

∘ □	1^+
∘ □	$1/n$
∘ □	$1/n$
∘ □	$1/n$
∘ □	$1/n$
□	$1/n$

$$OPT = 1^+ + \frac{n-1}{n} = 2 - \frac{1}{n}$$

$$Eq = 1^+$$

$$\boxed{PoA = 2 - \frac{1}{n} \to 2.}$$