- At this point it's natural to ask about *sequences* of infinitely many things and how one can reason about them. We'll start in a fairly intuitive fashion, and then formalise things later.

- Restricting our attention for the moment to the domain $\mathbb{R}$ of real numbers, we could define a sequence $(a_n)_{n\geq 1} = a_1, a_2, a_3, \ldots, a_n, \ldots.$ for $n \in \mathbb{N}^*$ *via* some rule. Often we'll write simply $(a_n)$, and may leave it to the context to know if the sequence is starting at *0* or *1*, or even some other number. For example:*

    - $a_n = n^2$

    - $a_1 = 1$, and $a_n = n \times a_{n-1} \ \forall\ n > 1$

    - $a_1 = 1,\ a_2 = 1,$ and $a_n = a_{n-1} + a_{n-2} \ \forall\ n > 2$

- In some sense, such sequences are ordered ∞–tuples, although that seems a tad tricky to see how to formalise. Perhaps better would be to define a function $\alpha : \mathbb{N} \longrightarrow \mathbb{R}$ so that $\alpha(n) = a_n$ thus in a formal sense making $a_n$ into "the *n*-th term" via the $n \in \mathbb{N}$ that it came from.

- If we were to *define* factorials by saying that *n!* were to be the product of all the integers from *1* up to and including *n*, then it would seem that to *prove* that the second rule above created the factorials would need infinitely many statements, and that's distinctly unpleasant! Instead, we take our cue from the way we constructed $\mathbb{N}$ in the first place. Intuitively ....

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* The first one is a sequence of squares, the second of factorials, and the last is the Fibonacci sequence.

- We could prove this in an intuitively rigorous inductive way by

    ① Remark that $a_1 = 1 = (1)!$

    ② Notice that *if* we were to assume that

    $$a_n = n!$$

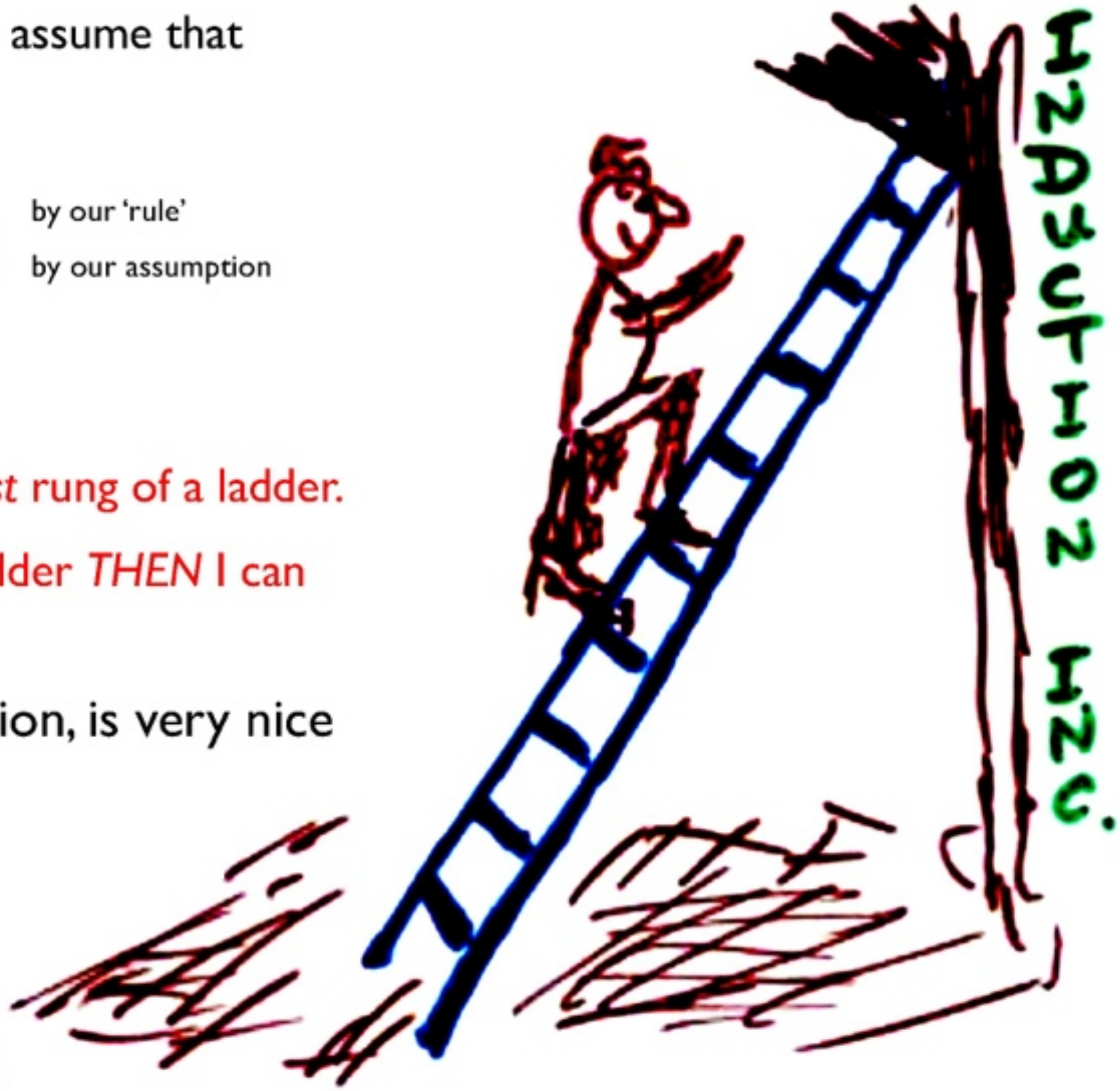
    then

    $$\begin{aligned} a_{n+1} &= (n+1) \times a_n && \text{by our 'rule'} \\ &= (n+1) \times (n!) && \text{by our assumption} \\ &= (n+1)! \end{aligned}$$

This is rather like saying

① I *can* put my foot on the *first* rung of a ladder.

② *IF* I'm on *any* rung of the ladder *THEN* I can step onto the next rung.




This way of arguing, called induction, is very nice because

a. We don't have to do infinitely many steps.

b. It's "jolly obvious" that we've covered every case!

- So how is this made formal? Returning to our definition of $\mathbb{N}$ (slide 8), we could make the following observations*

(i) $0 \in \mathbb{N}$, where of course $0 = \varnothing$

(ii) $n \in \mathbb{N} \implies \sigma(n) \in \mathbb{N}$

(iii) if $A \subseteq \mathbb{N}$ and $0 \in A$, then $( n \in A \implies \sigma(n) \in A ) \implies A = \mathbb{N}$

(iv) $\sigma(n) \neq 0 \;\; \forall\, n \in \mathbb{N}$

(v) for $n, m \in \mathbb{N}$, $\sigma(n) = \sigma(m) \implies n = m$

- These are known as the *Peano* axioms for the natural numbers (actually due to *Dedekind*).** The one we've listed as (iii) is the *principle of induction*. Although these are *axioms* for numbers, since we already have sets, we can deduce some of these, in particular (iv)*** and (v).

- We'll play with (iii) explicitly:

(a) Lemma: No $n \in \mathbb{N}$ is a subset of any of its elements. Let P($n$) = "*n is not a subset of any of its elements*".

*Proof:* Let $S = \{ n \in \mathbb{N} \mid n \not\subseteq k \;\forall\, k \in n \} \subseteq \mathbb{N}$. Now $0 = \varnothing \implies 0 \in S$, so the P($0$) is true (*base case*). Now suppose P($n$) true, ie, suppose that $n \in S$ (*induction step*). Then $n \subseteq n \implies n \notin n \implies \sigma(n) \not\subseteq n$. Moreover, for any $t$ with $\sigma(n) \subseteq t$, then $n \subseteq t$, so $t \notin n$, so $\sigma(n)$ also can't be a subset of any element of $n$, hence $\sigma(n)$ can't be a subset of any element of $\sigma(n)$, so $\sigma(n) \in S$, and P($\sigma(n)$) is true. So by (iii) we have that $S = \mathbb{N}$.

(b) Lemma: $\forall\, n \in \mathbb{N}$, every element of $n$ is a subset of $n$. The proof is similar, though with $S = \{ n \in \mathbb{N} \mid k \subseteq n \;\forall\, k \in n \}$.

(c) Suppose $n, m \in \mathbb{N}$ with $\sigma(n) = \sigma(m)$. So $n \in \sigma(n) \implies n \in \sigma(m) \implies (n \in m) \vee (n = m)$. Similarly, $(m \in n) \vee (m = n)$. Hence if $n \neq m$, then $(n \in m) \wedge (m \in n)$, but then lemma (b) $\implies (n \subseteq m) \wedge (m \subseteq n) \implies n = m$ ... oops! Hence $n = m$, and we've proven (v).

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* Recall that $\sigma(n) = n \cup \{n\}$, so $n$ is a set. Intuitively we think of $\sigma(n)$ as being represented by the expression $n+1$.

*** Actually, (iv) is very easy to show, so is left as an exercise for you.

** Dedekind, R. (1888). *Was sind und was sollen die Zahlen?* Vieweg, Braunschweig. Reprinted in R. Fricke, E. Noether & Ö. Ore (Eds.), *Gesammelte Mathematische Werke* (vol. 3, pp. 335-91). New York: Chelsea Publishing Company. 1969. Also see Peano, G., 1889, *Arithmetices Principia*, Bocca, Turin.