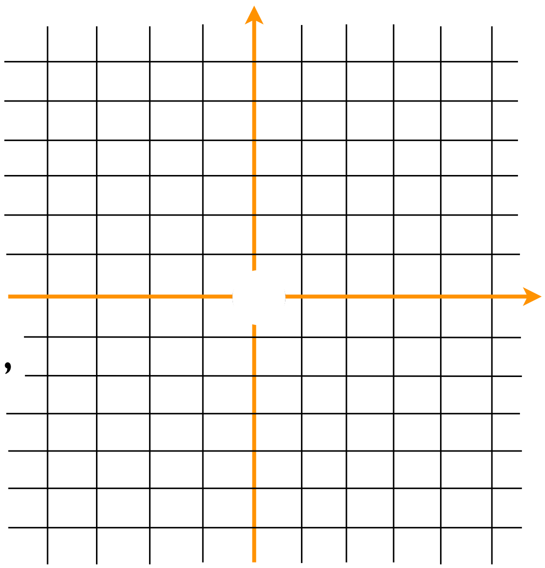
- Given two sets A , B , how do we decide which is the larger? We could either count the number of things in each and compare results, or simply pair up the terms (a , b) with $a \in$ A and $b \in$ B and see which set runs out first! Attempting to create such a *1-1* correspondence between A and B is really trying to construct a *1-1* function from A onto (a subset) of B or the other way around. On the other hand, trying to *count* the number of elements in a set S is really building a *1-1* function from S into $\mathbb{N}_{>0}$. We'll denote the *size* (or *cardinality*) of the set S by |S|.

- Defining the size for finite sets is easy, but amusing things happen when we start working with infinite sets. For example, which is the larger of $\mathbb{Z}$ and $2\mathbb{Z}$, where the latter is just the integer multiples of 2? Since $2\mathbb{Z} \subseteq \mathbb{Z}$ and $2\mathbb{Z} \neq \mathbb{Z}$ it's obvious that $|2\mathbb{Z}| \leq |\mathbb{Z}|$, where we're inclined to treat the inequality as strict. However, consider the function $f : \mathbb{Z} \longrightarrow 2\mathbb{Z}$ given by $f(n) = 2n$. It's easy to show that f is *1-1*, which means that $|\mathbb{Z}| \leq |2\mathbb{Z}|$ and hence $|2\mathbb{Z}| = |\mathbb{Z}|$. Similarly, we can show that $|\mathbb{Z}| = |\mathbb{N}|$. This ability for a set to be in *1-1* correspondence with a *proper* subset of itself could be taken as a defining property of infinite sets.

- Let's compare the sizes of $\mathbb{Z}$ and $\mathbb{Z}^2$. The function $f : \mathbb{Z} \longrightarrow \mathbb{Z}^2$ given by $f(n) = (n , 0)$ is *1-1* (check this), so $|\mathbb{Z}| \leq |\mathbb{Z}^2|$. Consider the function $g : \mathbb{Z}^2 \longrightarrow \mathbb{N}$ as described in the picture. This is also *1-1*, so $|\mathbb{Z}^2| \leq |\mathbb{N}| \leq |\mathbb{Z}|$, and hence $|\mathbb{Z}^2| = |\mathbb{Z}|$. Notice that since $\mathbb{Z} \subseteq Q \subseteq \mathbb{Z}^2$ (by the construction of Q), we have $|\mathbb{Z}| \leq |Q| \leq |\mathbb{Z}^2|$, and hence $|Q| = |\mathbb{Z}|$. We say that a set S is *countable* if it has the same size as $\mathbb{N}$, and write $|S| = \aleph_0$ (using the Hebrew letter *aleph*).



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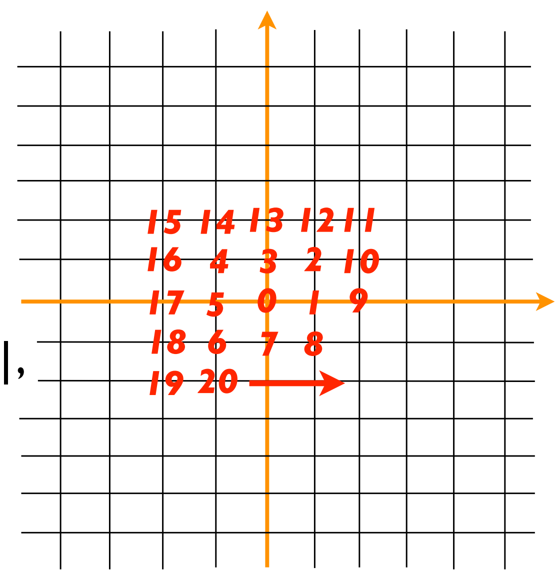

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- The argument proceeds by contradiction, starting by supposing $\mathbb{R}$ to be countable.

  - It will be enough to consider the subset* $[0\,,1] \subseteq \mathbb{R}$; since if $\mathbb{R}$ is countable then every subset of $\mathbb{R}$ must be countable, consequently there would be an explicit one-to-one function from $[0\,,1]$ to $\mathbb{N}$. Looking at this list, where we have written each number in $[0\,,1]$ as an infinite decimal, we will now exhibit a number which fails to be in that supposedly complete list.

$$
\begin{array}{lcll}
\alpha_1 & = & 0\cdot a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}\,\ldots\ldots & \mathbf{1} \\
\alpha_2 & = & 0\cdot a_{21}a_{22}a_{23}a_{24}a_{25}a_{26}\,\ldots\ldots & \mathbf{2} \\
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\alpha_6 & = & 0\cdot a_{61}a_{62}a_{63}a_{64}a_{65}a_{66}\,\ldots\ldots & \mathbf{6} \\
\downarrow & & \downarrow & \downarrow
\end{array}
$$

  - Consider the number $\beta = 0\cdot b_1\, b_2\, b_3\, b_4\, b_5\, b_6\, \ldots.$ where the value of the *r*-th decimal place is given by $b_r = a_{rr}$ (mod 5) so that, for example, if $a_{rr} = 8$ then $b_r = 3$. Clearly $\beta \neq \alpha_1$ since it's about as wrong as possible in the first decimal place. Similarly, $\beta \neq \alpha_2$ since it fails to match at the second decimal place. Indeed, β fails to equal *any* of the α_r, always failing in the 'diagonal position', which means that β fails to be in the supposedly complete list of reals in $[0\,,1]$. Since this argument can be repeated for any purported *counting* of $[0\,,1]$, this proves that $[1\,,0]$, and hence also $\mathbb{R}$, is uncountable.

- Another version of this proof applies nicely to sets and their power sets. Claim $|A| \lneqq |\mathscr{P}(A)|$.

  - Consider $f : A \longrightarrow \mathscr{P}(A)$ given by $f(a) = \{a\}$. Clearly this is 1-1, so $|A| \leq |\mathscr{P}(A)|$. To prove the inequality strict, suppose false, i.e., suppose $\exists$ surjection $g : A \longrightarrow \mathscr{P}(A)$. Let's define $B = \{\, a \in A \mid a \notin g(a) \,\}$, then certainly $B \subseteq A$, so $B \in \mathscr{P}(A)$, so since g is onto, $\exists\, a' \in A$ with $g(a') = B$. If $a' \in B$, then $a' \notin g(a') = B$ ... oops! But if $a' \notin B$, then $a' \in g(a') = B$ ... oops again!! So our original supposition is untenable, and hence the inequality must be strict.

- This *uncountable* set has size $\aleph_1$, the notation for which probably hints to you that this is the start of a sequence of distinct infinite sizes!!!!

* The notation $(a\,,b) = \{\, x \in \mathbb{R} \mid a < x < b \,\}$ and $[a\,,b] = \{\, x \in \mathbb{R} \mid a \leq x \leq b \,\}$ for *open* and (respectively) *closed intervals* is standard, with obvious meanings for $[a\,,b)$ and $(a\,,b]$.

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