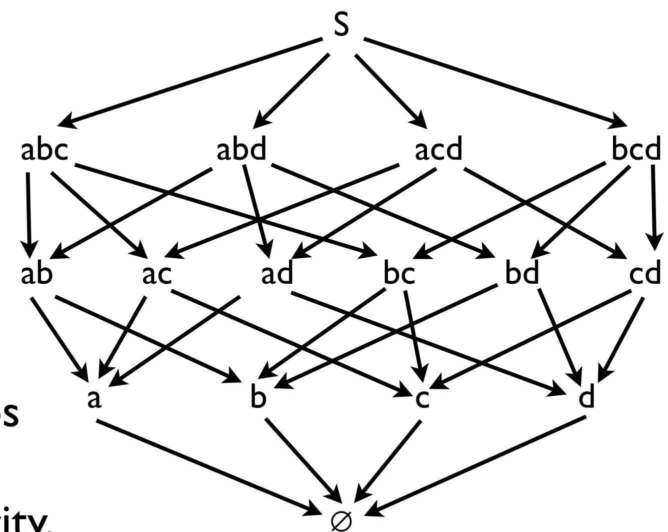
- Let's consider other potential relationships within a set. A *preorder* on a set S is a relation $\leq$ which is both *reflexive* and *transitive*, so $a \leq a \ \forall\ a \in S$ and $(a \leq b) \wedge (b \leq c) \Longleftrightarrow a \leq c$.

- Just as the archetype for the idea of equivalence relations is '=', so the primary motivator for the concept of order relations is '$\leq$'.* In particular, a *partial order* on a set S is a preorder which is also *antisymmetric*, namely $(a \leq b) \wedge (b \leq a) \Longrightarrow a = b$.** Note that a preorder on S can be made into a partial order on S/~ via the equivalence relation $s \sim t \Longleftrightarrow (s \leq t) \wedge (t \leq s)$.

- A set equipped with a partial order is called a *poset* (or *partially ordered set*). If a poset also has the property that $s, t \in S \Longrightarrow (s \leq t) \vee (t \leq s)$ *** then the order is called a *total order* and the set is often called a *chain* (or *linearly ordered set*).****

- The reason for the name 'chain' for a total order becomes clear as soon as we start drawing diagrams to represent such relationships. Given a (non-empty) set S, if we build a set comprising *all* the subsets of S (including ∅ and S itself), then we call this the *power set* of S and denote it by $\mathscr{P}(S)$, though 2^S is used by some authors.***** In the diagram, we've indicated the relationships within the power set of $S = \{a, b, c, d\}$ under the partial order $\subseteq$, notating, e.g., the subset $\{a, b, d\}$ by abd for clarity.



* We have to be careful here not to fall into the trap of seeing a familiar symbol and ascribing to it a meaning that's yet to exist. In particular, in the sequel we'll use the $\leq$ sign for potential partial orders, but cannot presume that it has any of the familiar properties of 'less than or equals' until proven to do so.

** Some authors use 'partial order' only for *irreflexive* orders. It's safer to state that a given partial order is *weak* or *strict* depending on whether or not equality is allowed. We will assume partial orders to be weak unless stated otherwise.

*** Notice that this means that *every* pair of elements of S are related, ergo the label *total*.

**** As helpful examples, consider $\mathbb{Z}$. This is totally ordered with the usual meaning of $\leq$, however it's hard to see how to construct a natural total order on points in the plane, even though there's a multitude of reasonable partial orders which make sense there.

***** If you calculate the size of the power set of any finite set, it'll become immediately clear what the motivation for this particulate notation is.

- In the previous diagrammed example, there were explicit maximum and minimum elements. In general this cannot be presumed ... indeed there may be many maximal and minimal elements in a given poset. More formally, the element m is *minimal* in a poset S iff $s \leq m \Longrightarrow s = m$,* and is the *minimum* in S iff $m \leq s \; \forall \; s \in$ S. Similar definitions hold for *maximal* and *maximum*.** Things can become quite interesting when we consider infinite sets!

- We define a poset to be *well-ordered* if every non-empty subset has a minimum.*** Notice that a well-ordered poset S must be a chain, since for $a, b \in$ S, the set $\{a , b\}$ is a non-empty subset, so must have a minimum, and hence either $a \leq b$ or $b \leq a$.

- If S is a poset, then a subset T ⊆ S is said to have an *upper bound* in S if $\exists \; u \in$ S such that $t \leq u \; \forall \; t \in$ T . Notice that we're not requiring the potential upper bound actually to reside in T itself.**** Suppose we have a potentially infinite poset S with the additional property that every non-empty chain C in S has an upper bound in S. Can we guarantee that S has at least one maximal element? (*Although this seems very reasonable, it can't be proven from the usual axioms of set theory ... so in order to claim the existence of such a maximal element we have to take this as a fresh axiom! It goes by the name of Zorn's Lemma, and is fundamental in dealing with infinite ordered sets. It has various equivalent forms, the most common being the Axiom of Choice.*)

- A final definition before we move on to considering what infinity might be ... If A and B are posets, each with their own order relations, then a function f : A ⟶ B is *order-preserving* if $x \leq_A y \iff \mathrm{f}(x) \leq_B \mathrm{f}(y) \; \forall \; x, y \in$ A. The definition of *order-reversing* for a function is natural.

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* Notice that the definition of m being minimal only requires that there be *nothing related* to m under the partial order which is smaller than m.

** A nice illustrative example for you to construct would be to let S be the integers from *2* to *48* inclusive, with the order relation p ≤ q ⇔ p | q . Notice also that if we were to consider the real numbers *strictly* between *2* and *10* with the usual ordering, then that set has *no* maximal or minimal elements.

*** Notice that with the usual meaning of ≤, the set $\mathbb{N}$ is well-ordered but that both $\mathbb{Z}$ and the *strictly positive* reals are not.

**** Nor are we requiring that u be the *least* upper bound ... it's permitted to be quite generous if desired! We can make similar definitions for *lower bounds*. If they exist, and if we want to refer to the *least* upper bound (or the *greatest* lower bound), then the standard terminology is *supremum* (and *infimum*).