- Now that we have sets to play with, it's time to consider how to define *functions* on these sets. We do this for a pair of sets A and B by first defining a *graph of a mapping from* A *to* B to be a relation $\mathcal{g} \subseteq A \times B$ such that $(a, s) \in \mathcal{g} \wedge (a, t) \in \mathcal{g} \implies s = t$.* This means that there's no ambiguity in knowing where each element of A gets mapped to.

- The *mapping* itself is notated as $f : A \longrightarrow B$ with $f(a) \in B$ being defined by $(a, f(a)) \in \mathcal{g}$ whenever that pair exists in $\mathcal{g}$. If $\forall a \in A\ \exists b \in B$ with $(a, b) \in \mathcal{g}$, then the mapping f is said to be a *function*.** The set A is then the *domain* of f, and we define the *image* of f to be the set $f(A) = \{ y \in B \mid \exists x \in A \text{ with } f(x) = y \}$, sometimes called the *co-domain* or *range* of f. Often a mapping f can be made into a function by *restricting* it to a subset $S \subseteq A$, notation for which is $f|_S$, e.g., the mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ with $f(x) = \sqrt{x}$ becomes a function as $f|_{x \geq 0}$ or $f|_{x \geq 0}$.

- Some further refinements:

  (i) If $b \in B \implies \exists a \in A$ with $(a, b) \in \mathcal{g}$ then f is *onto* B (or *surjection* or *epimorphism*).

  (ii) If $(s, b) \in \mathcal{g} \wedge (t, b) \in \mathcal{g} \implies s = t$ then f is *one-to-one* (or *injection* or *monomorphism*).

  (iii) If f satisfies (i) and (ii) above, then it is a *bijection* (or *isomorphism*).***

  (iv) If $f : A \longrightarrow A$ then f is often called an *endomorphism*.

  (v) If f is both an endomorphism and a bijection, then it's often called an *automorphism*.***

* This is essentially the flip side of the definition we'll see shortly of a function being one-to-one. Notice that in this context, taking a number to its square is a mapping, whereas taking a number to its square root fails to be a mapping. Not requiring this restriction yields *multi-valued* mappings and functions, We can remove the multi-valuedness by restricting the range.

** So that everything in A is related to something in B. Colloquially this means that everything in A gets mapped *somewhere* by the function. Often we'll use the term *morphism* (or *homomorphism*) for functions which preserve all the available structure ... in this case the only structure is that of a set.

*** It's worth observing that these one-to-one and onto functions f are really valuable. Essentially, in these cases f provides an ideal dictionary which shows that A and B are in essence 'the same' by giving a 'translation' from the labels in A to the labels in B. This applies not only in the relatively naïve context of set theory, but also in the rather deeper algebraic, geometric, analytic, and even logical areas. This places a significant premium on the ability to construct such bijections from any given arbitrary function. Such techniques (often exploiting equivalence relations to clean things up) enable us to 'translate' from rather more complex environments to simpler ones. We'll explore various standard ways of doing this as we proceed through the course.

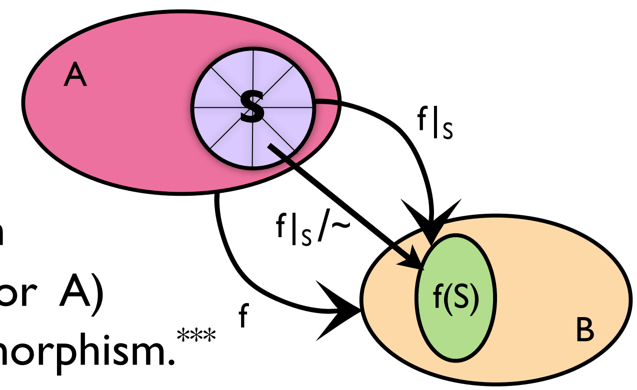
- Now that we have functions, we can play the usual game of exploring how to combine them!

- If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are mappings, and (*by abuse of notation*) writing $g = g\,|_{f(A)}$, then the *composite* mapping is $g \circ f : A \longrightarrow C$, and is defined by $g \circ f\,(a) = g(\,f(a)\,)$. Diagrams such as this are often used to describe collections of interacting mappings, and are called *commutative diagrams*, since the same values are obtained by taking alternative routes. The sport of checking values by following such diagrams is called *diagram chasing* !!

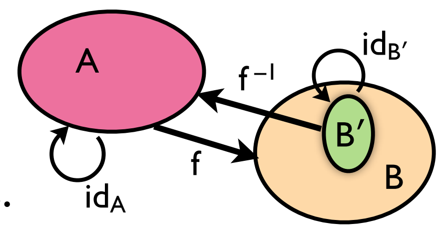
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

- We define the *pre-image* of $Y \subseteq B$ under f to be the set $f^{-1}(Y) = \{\, x \in A \mid f(x) \in Y \,\}$. Do note that although we've used the notation f^{-1}, this absolutely *does not* presume the existence of any inverse function or mapping!!!!

- Notice that if $f : A \longrightarrow B$ is a mapping, and if S is the largest subset of A for which $S \times f(S) \subseteq \mathcal{g}$, then $f\,|_S$ is a function.* *By abuse of notation*, denoting $f\,|_S : S \longrightarrow f(S)$ by f, the function f has become an epimorphism.** By defining a relation on S (or A) $a_1 \sim a_2 \iff f(a_1) = f(a_2)$ we make $f : S/{\sim} \longrightarrow f(S)$ into an isomorphism.***



- If the function $f : A \longrightarrow B$ is one-to-one, then an actual *inverse function* $f^{-1} : B' \longrightarrow A$ will exist, where $B' = f(A)$, since there will be a well-defined value for $f^{-1}(b)$ for each $b \in B'$. Then the composite functions $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_{B'}$ yield the corresponding identity functions.



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* What we've done here is simply to dump the part of A that f doesn't map from, so that the set S is the *de facto* domain of f. Of course, S doesn't have to be a *largest* subset, but it makes sense to beef it up as much as possible.

** Now we've thrown away the part of B that f couldn't reach, thus making it a surjection (onto).

*** By constructing this equivalence relation (*you should check that it is one!*) we've *divided out* the differences in the domain, thus making f an injection (one-to-one), and hence now a bijection. Thus $S/{\sim}$ and $f(S)$ are essentially the same, with explicit translation being provided by f. We'll elaborate on this process later in the course when we're dealing with algebra (*at this point, in the very simple context of sets, it will seem rather artificial to have gone through such shenanigans*).