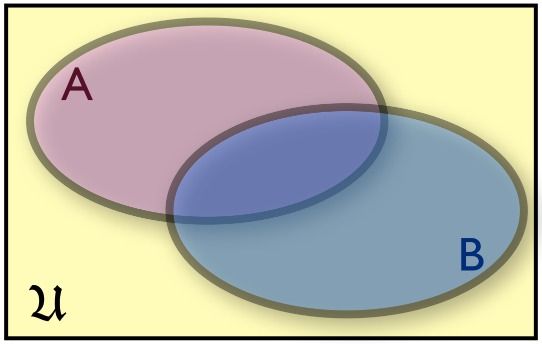
- There are a lot of niceties in defining what a *set* should be, but for now we'll leave that to the more esoteric regions of the foundations of mathematics, and be satisfied with the inherently problematic definition that a set is a collection of objects defined by some rule (i.e., the rule tells us whether any given objects should or should not be in the collection).*

- Unless we say otherwise, we'll treat sets like $\{a, b, c, a, b, a, d\}$ and $\{a, b, d, c\}$ as being equal, i.e., the order of listing members is irrelevant, and any repetition of the same element in the description doesn't add in multiple copies of it. **

- As in our discussion of logic, so again for sets we'll start by looking at ways of manipulating sets.*** There are two particular sets of importance: *the empty set*, denoted $\varnothing$, and *the universe* (comprising everything under potential consideration, which *we'll* choose to denote by $\mathfrak{U}$).

- There are some natural ways of acting on sets:
  - If A and B are two sets, then A − B is the set of all things which are in A but not in B and could write this formally as $A - B = \{ x \in A \mid x \notin B \}$, the *set difference*. Note that $x \in A$ denotes that x is a member of the set A, and the vertical line in this context means 'such that'.
  - $A \cup B$ is the *union* of A and B, formally $A \cup B = \{ x \in \mathfrak{U} \mid x \in A \vee x \in B \}$.
  - $A \cap B$ is the *intersection of* A and B, formally $A \cap B = \{ x \in \mathfrak{U} \mid x \in A \wedge x \in B \}$.
  - A^c is the *complement* of A, formally $A^c = \mathfrak{U} - A$.
  - A + B is the *symmetric difference* of A and B, formally $A + B = (A - B) \cup (B - A)$.



* For an elaboration of the problems, look up Russell's paradox, and think in terms of such collections being able to be a bit too large ... the formal ways of fixing this are essentially ways of ensuring that there's a restriction on how big a set can be -- infinite is perfectly fine however.

** In the case that this latter aspect is permitted, we'll be explicit in calling such animals *multisets*.

*** It's worth noting that whenever any somewhat general results are stated, it's often helpful to have a few specific examples to play with so that you can start to get a feel for what the results mean in practice. *Proof by example* is a common attempt by many students, but only works if you test it for every conceivable example(!!!), but it is nevertheless a great way to help decide if you actually *believe* the result.

- We write $A \subseteq B$ to mean that A is a *subset* of B, meaning that everything in A is also in B. Note that for two sets A and B to be equal means that $A \subseteq B$ and $B \subseteq A$.

- There are a number of quasi-algebraic manipulations we can do with sets: *

1. $A \cup (B \cup C) = (A \cup B) \cup C$
2. $A \cup B = B \cup A$
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
4. $A \cup \varnothing = A$
5. $A \cup A^c = \mathfrak{U}$
6. $A \cup B = A$ for all sets $A \Rightarrow B = \varnothing$
7. $A \cup B = \mathfrak{U}$ and $A \cap B = \varnothing \Rightarrow B = A^c$
8. $(A^c)^c = A$
9. $\varnothing^c = \mathfrak{U}$
10. $A \cup A = A$
11. $A \cup \mathfrak{U} = \mathfrak{U}$
12. $A \cup (A \cap B) = A$
13. $(A \cup B)^c = A^c \cap B^c$ **

Sample wordy proof for 3:

Let $x \in (A \cup B) \cap (A \cup C)$, then by definition of $\cap$ this means that $x \in A \cup B$ *and* $x \in A \cup C$.
$x \in A \cup B$ means by definition of $\cup$ that $x \in A$ *or* $x \in B$, *or both.*
$x \in A \cup C$ means by definition of $\cup$ that $x \in A$ *or* $x \in C$, *or both.*
So if $x \notin A$ *then* it has to be that $x \in B$ *and* $x \in C$.
If $x \in A$ *then* it needn't be in B or C, but no problem occurs if it is.
Hence $x \in A \cup (B \cap C)$, and we've shown that
$A \cup (B \cap C) \supseteq (A \cup B) \cap (A \cup C)$.

To show the converse, let's argue by contradiction.
Suppose that $y \in A \cup (B \cap C)$ but that $y \notin (A \cup B) \cap (A \cup C)$. (*)
$y \in A \cup (B \cap C)$ means by definition of $\cup$ that $y \in A$ *or* $y \in B \cap C$, *or both.*
If $y \in A$ then $y \in A \cup B$ *and* $y \in A \cup C$, which would contradict (*).
So $y \notin A$.
Hence $y \in B \cap C$, which by definition of $\cap$ means that $y \in B$ *and* $y \in C$.
But then $y \in A \cup B$ *and* $y \in A \cup C$, again contradicting (*).
Hence (*) is false, and we've shown that
$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Thus we have that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Sample symbolic proof for 3:

$x \in (A \cup B) \cap (A \cup C) \iff (x \in A \cup B) \wedge (x \in A \cup C)$, by definition of $\cap$
$\iff ((x \in A) \vee (x \in B)) \wedge ((x \in A) \vee (x \in C))$, by definition of $\cup$
$\iff ((x \in A) \wedge (x \in A)) \vee ((x \in B) \wedge (x \in C))$, by distribution of $\wedge$ and $\vee$
$\iff ((x \in A) \vee (x \in B \cap C))$, by definition of $\cap$
$\iff x \in A \cup (B \cap C)$, by definition of $\cup$

- All of the above have *dual* versions obtained by swapping $\cup$ and $\cap$, and $\varnothing$ and $\mathfrak{U}$.

* You might like to try proving them.

** This is one of the De Morgan's laws ... the other is its dual.