An Analysis of Troubled Assets Reverse Auction

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Abstract. In this paper we study the Nash-equilibrium and equilibrium bidding strategies of the Pooled Reverse Auction for troubled assets. The auction was described in (Ausubel & Cramton 2008[1]). We further extend our analysis to a more general class of games which we call *Summation Games*. We prove the existence and uniqueness of a Nash-equilibrium in these games when the utility functions satisfy a certain condition. We also give an efficient way to compute the Nash-equilibrium of these games. We show that then Nash-equilibrium of these games can be computed using an ascending auction. The aforementioned reverse auction can be expressed as a special instance of such a game. We also, show that even a more general version of the well-known oligopoly game of Cournot can be expressed in our model and all of the previously mentioned results apply to that as well.

1 Introduction

In this paper, we primarily study the equilibrium strategies of the pooled reverse auction for troubled assets which was described in [1]. The US Treasury is purchasing the troubled assets to infuse liquidity into the market to recover from the current financial crisis. Reverse auctions in general have been a powerful tool for injecting liquidity into the market in places where it will be most useful. As explained in [1] a simple and naive approach for the government could be to run a single reverse auction for all the holders of toxic assets as follows. The auctioneer(government) then sets a total budget to be spent. The auctioneer starts at a price like 100¢ on a dollar. All the holders, bid the quantity of their shares that they are willing to sell at the current prices. There can be excess supply. The auctioneer then lowers the price in steps e.g. 95¢, 90¢, etc. and bidders indicate the quantities that they are willing to sell at each price. At some point (for example at 30¢ on a dollar) the total supply offered by all the holders for sale equals or falls bellow the specified budget of the treasury. At that point the auction concludes and the auctioneer buys the securities offered at the clearing price. As explained in [1], this simple approach is flawed as it leads to a severe *adverse selection problem*. Note that at the clearing price the securities that are offered are only the ones that are actually worth less than 30¢ on each dollar of face

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value. They could as well worth far bellow 30¢. In other words, the government would pay most of its budget to buy the worst of the securities.

In [1], the authors propose the following two type of auctions.

- A Security by Security Reverse Auction
- A Pooled Reverse Auction

They are both part of a two phase plan. The first one can be used to extract private information of holders about the true value of the securities to give an estimate on how much each security and similar securities are actual worth of. Later, that information can be used to establish reference prices in the Pooled Reverse Auction.

In this paper we focus our attention on the second class of auction. In a Pooled Reverse Auction, different securities are pooled together. The government puts a reference price on each security and then runs a reverse auction on all of them together. We explain this auction in more detail in section 2.

In section 3, we study the Nash-equilibrium and the bidding strategies of the Pooled Reverse Auctions in detail. We then create a more abstract model of it at the end of section 2. In section 4 we describe a general class of games that can be used to model the Pooled Reverse Auction as well as other problems. In section 4, we give some exciting result on these games. We give a condition which is sufficient for the existence of a Nash-equilibrium. We further explain how the Nash-equilibrium can be computed efficiently using a an ascending auction-like mechanism. Later in section 5, we show how we can apply our result of section 4 to Pooled Reverse Auctions. section 6 explains how a more general version of the Cournot's oligopoly game can be expressed in our model.

1.1 Related Work

We partition the related works to two main groups. The first group that is closely related to our model, are computing equilibrium in Cournot and public good provision games. The second one with similar model but different objective are the works related to bandwidth sharing problems and the efficiency of computed equilibria.

One well known problem that can be considered as an example of our model is the Cournot's oligopoly game. It can be described as an oligopoly of firms producing a homogeneous good. The strategy of firm i is to choose q_i which is the quantity it produces. Assuming that the production cost is c_i per item, the utility of firm i is $(p(Q) - c_i)q_i$ for which $Q = \sum_i q_i$ is the total production and $p(Q)$ is the global price of the good based on the total production. There is a vast amount of literature on Cournot games (e.g. [7]). Different aspect of Cournot equilibrium has been studied (For example, in [3] Bergstrom and Varian, studied the effect of taxation on Cournot equilibrium and also showed some characteristics of the Cournot equilibrium.)

Another set of results, with similar model, but with different criteria are the works related to bandwidth sharing problem. At a high level, the problem is to allocate a fixed amount of an infinitely divisible good among rational competing users. [8] studies this problem from pricing perspective. Kelly [6], considered a generalized variant of this problem in the context of routing and charging (However the equilibrium point of his mechanism was not fully efficient) His model, for a single resource with fixed supply, is to give each person proportional to his bid from the resource and charge him his bid. Later, Johari et al in [4], showed that Kelly's mechanism is at least 75% efficient at the equilibrium point. In another work, Johari et al show that, Kelly's model minimizes efficiency loss (at the equilibrium point) when price discrimination is not allowed and then they present a class of mechanisms that has an efficient outcome at the equilibrium point assuming that price discrimination is allowed ([5]).

2 Model for Pooled Reverse Auction

In this section, we explain the basic model for the reverse auction of pooled securities. We will use this model throughout the rest of this paper. We start by explaining our notations:

- There are n bidders $N = \{1, \cdots, n\}$, and m securities.
- Government has evaluated a reference price of r_j for each security j. Also let $\mathbf{r} = (r_1, \cdots, r_m)$ denote the vector of reference prices for all the m securities. The reference prices are public information. These prices are in the form of the ratio of the evaluated price to the face price and are expressed in cents per dollar. For example $r_j = 0.75$ means every dollar of the face value of the security is actually worth 75¢.
- Each bidder i holds $\bar{q}_{i,j}$ shares of security j. Also let $\bar{\mathbf{q}}_i = (\bar{q}_{i,1}, \cdots, \bar{q}_{i,m})$ denote the vector of the quantities of shares that bidder i holds from each security. The shares are expressed in quantity of the face value.
- Each bidder has a private valuation function $v_i(l)$ for receiving a liquidity amount of l. In a quasi-linear setting, we would assume that $v_i(l) = l$. In our model, we assume the v_i could be an arbitrary function. v_i can capture the bonus for acquiring a needed amount of liquidity or can be negative to account for the cost incurred by the shortage thereof. For example consider the following:

$$v_i(l) = \begin{cases} l + (l - L_i) & l \leq L_i \\ l & l \geq L_i \end{cases} \tag{2.1}$$

  We could interpret the above v_i as the following. Bidder i has a liquidity need of L_i dollars. She incurs a cost of $L_i - l$ dollars if she raises only l dollars where $l < L_i$. Her value for any liquidity that she receives beyond L_i is just the same as the amount that she receives. The experimental study of reverse auction for troubled assets in [2] considers two cases for v_i. In the first case, each bidder i has a liquidity need L_i and $v_i(l) = 2l$ for $l \leq L_i$ and $v_i(l) = l + L_i$ for $l > L_i$. In the second case, bidders don't have liquidity needs, so $v_i(l) = l$. In this paper, we consider arbitrary v_i under some constraints as we will see later.

– Each bidder i has a private value of $w_{i,j}$ for each dollar of security j. Also let $\mathbf{w}_i = (w_{i,1}, \cdots, w_{i,m})$ denote the vector of the valuations of bidder i for different securities. In reality, we should have assumed a single common value for each security which is unknown and can only be computed by aggregating all the private information of all bidders. However, that model is prohibitively hard to analyze in case of non trivial valuation functions for liquidity (i.e. when $v_i(l)$ is not the identity function). Therefore, we assume that $\mathbf{w}_i$ is the private values of bidder i for the securities.

Next, we briefly explain the reverse auction mechanism for pooled securities as described in [1].

Auction 1 (Pooled Reverse Auction) *Initially, the auctioneer (government) establishes the reference prices for all the securities. These reference prices are supposed to be the best estimate of the government about the true value of the securities. The reference prices are announced publicly.*

The auction uses a single descending clock α which specifies the current prices as a percentage of the reference prices. For example, $\alpha = 110\%$ means the current price of each security is 110% of its reference price. As the clock goes down, participants update their bids. Bidder i submits a bid $\mathbf{b}_i = (b_{i,1}, \cdots, b_{i,m})$, where $b_{i,j}$ is the quantity of shares from security j that bidder i would like to sell at the current prices. These quantities are specified in terms of dollars of face value. The auctioneer collects all the bids and computes the activity points *for each bidder i as $a_i = \mathbf{r} \cdot \mathbf{b}_i$ (remember $\mathbf{r}$ is the vector of reference prices) . In other words, the activity points of each bidder is her bid quantity for each security times the reference price of that security summed over all the securities. The auctioneer also computes the total activity point $A = \sum_i a_i$. Assuming that M is the total budget of the government, the clock keeps going down for as long as $A\alpha > M$. In practice, the clock goes down in discrete steps. At each step the auctioneer collects all the bids and computes the aggregate activity point. At the first step that $A\alpha$ becomes less than or equal to M, the clock stops and the auction concludes. The auctioneer then buys from each bidder the quantity of shares specified in her bid. Bidders are paid at the current prices (i.e. the reference price scaled by the current value of the clock). Assuming that α^* was the final value of the clock and for each bidder i, $\mathbf{b}_i^*$ was the final bid of bidder i, the amount of liquidity that bidder i receives is $\alpha^* \mathbf{r}_i \cdot \mathbf{b}_i^*$.*

In the next section we study the equilibrium of the above auction.

3 The Equilibrium of Pooled Reverse Auction

In this section, we study the Nash-equilibrium of Auction. 1 and propose a method that can be used to efficiently compute that. We also develop a bidding strategy that leads to the Nash-equilibrium.

First, we show how to compute the utility of each bidder i. Assume that $\mathbf{b}_i$ is the bid of bidder i and α is the current value of the clock. Also, as we defined in section 2, $v_i(l)$ is the valuation of bidder i for receiving amount l of liquidity and $\mathbf{w}_i = (w_{i,1}, \cdots, w_i)$ is the vector of her valuations for different securities. We denote by u_i, the tentative utility of bidder i which is her utility if the auction stops at the current value of the clock. u_i can be computed as the following:

$$u_i = v_i(\alpha \mathbf{r} \cdot \mathbf{b}_i) - \mathbf{w}_i \cdot \mathbf{b}_i \tag{3.1}$$

Before we start with the bidding strategies, we restate some of the definitions from Auction. 1.

– For a bidder i with current bid $\mathbf{b}_i$, we use a_i to define her activity point which is defined as:

$$a_i = \mathbf{r} \cdot \mathbf{b}_i \tag{3.2}$$

– The total activity point of all bidders is defined as:

$$A = \sum_{i=1}^{n} a_i \tag{3.3}$$

– The auction clock, α, keeps going down for as long as $\alpha A > M$ where M is the total budget of the auctioneer. If we denote the value of the clock when the auction stops by α^*, then $\alpha^* A \leq M$. Note that, to simplify the analysis, we assume quantities do not need to be integers. We also assume that the clock changes continuously and bidders update their bids continuously as well. Respectively, we may assume that when the auction concludes, the auctioneers budget constraint is met with equality so:

$$\alpha^* A = M \tag{3.4}$$

Next, we show that the best strategy for each player i can be described by just specifying the activity points that she needs to generate. In other words, the only thing that bidder i has to decide is how much activity point to generate and her best bid vector can be specified as a function of that.

Lemma 1. *In order to play her best strategy, bidder i only needs to choose her activity points a_i and then among all the bid vectors $\mathbf{b}_i \in [\mathbf{0}, \bar{\mathbf{q}}_i]$[1] such that $\mathbf{r} \cdot \mathbf{b}_i = a_i$ her best strategy is to submit a bid $\mathbf{b}_i$ that minimizes $\mathbf{w}_i \cdot \mathbf{b}_i$. We will refer to one such bid vector as $\mathbf{b}_i(a_i)$. Formally:*

$$\mathbf{b}_i(a) = \text{argmin}_{\mathbf{b}}\, \mathbf{w}_i \cdot \mathbf{b} \;\;:\;\; \mathbf{b} \in [\mathbf{0}, \bar{\mathbf{q}}_i] \wedge \mathbf{r} \cdot \mathbf{b} = a \tag{3.5}$$

[1] We use the notation $[\mathbf{a}, \mathbf{b}]$ to denote all the vectors that are componentwise greater than or equal to $\mathbf{a}$ and less than or equal to $\mathbf{b}$

Proof. The only variable parameter in the auction that correlates the utility of different bidders is the the clock value α and the only way individual bidders affect that variable is through their activity points. If a bidder such as i keeps her activity points fixed and changes her bid vector, the outcome of the auction will not change. However, among all the bid vectors that generate the same amount of activity point, the one with the lowest $\mathbf{w}_i \cdot \mathbf{b}_i$ produces the highest utility for bidder i.

Based on Lemma 1 to describe a best strategy for a bidder i we only need to specify the activity points a_i that she should bid and then Lemma 1 tells us what condition the corresponding bid vector should satisfy. The next lemma describes how we can efficiently compute $\mathbf{b}_i(a_i)$ for any given a_i.

Lemma 2. *For any given $a_i \in [0, \mathbf{r} \cdot \bar{\mathbf{q}}_i]$ we can compute $\mathbf{b}_i(a_i)$ by using the following procedure.*

Without loss of generality, assume securities are sorted in decreasing order of $\frac{r_j}{w_{i,j}}$ so that $\frac{r_j}{w_{i,j}} \geq \frac{r_{j+1}}{w_{i,j+1}}$. To find the bid vector, we start from an initial zero bid vector and increase each $q_{i,j}$ up to $\bar{q}_{i,j}$ starting at $j = 1$ until the generated activity point reaches a_i. The following is a more formal definition of $\mathbf{b}_i(a)$:

$$\mathbf{b}_i(a) = (\bar{q}_{i,1}, \cdots, \bar{q}_{i,y-1}, b_{i,y}, 0, \cdots, 0) \tag{3.6}$$

such that:

$$r_y b_{i,y} + \sum_{j=1}^{y-1} r_j \bar{q}_{i,j} = a \tag{3.7}$$

Proof. The proof is by contradiction. Suppose bidder i is submitting a bid vector $\mathbf{b}_i$ which is not according to the mentioned schema but minimizes $\mathbf{w}_i \cdot \mathbf{b}_i$ subject to $\mathbf{b}_i \in [\mathbf{0}, \bar{\mathbf{q}}_i]$ and $\mathbf{r} \cdot \mathbf{b} = a_i$. So there should be two different securities j and k such that $\frac{r_j}{w_{i,j}} > \frac{r_k}{w_{i,k}}$ and in her bid vector $b_{i,j} < \bar{q}_{i,j}$ and $b_{i,k} > 0$. We argue that she can decrease $b_{i,k}$ by some $\epsilon > 0$ and increase $b_{i,j}$ by $\epsilon \frac{r_k}{r_j}$. Note that the change in her activity points is $-\epsilon r_k + \epsilon \frac{r_k}{r_j} r_j$ which is 0. However that decreases $\mathbf{w}_i \cdot \mathbf{b}_i$ by $\epsilon w_{i,k} - \epsilon \frac{r_k}{r_j} w_{i,j}$ which is always positive and contradicts our assumption about $\mathbf{w} \cdot \mathbf{b}_i$ being the minimum.

Intuitively, Lemma 2 is saying that a strategic bidder should never sell any shares of a security j unless for any other security k for which $\frac{r_k}{w_{i,k}} > \frac{r_j}{w_{i,j}}$ she has already sold all of her shares of security k.

Definition 1. *We can define a cost function $c_i(a) : [0, \mathbf{w}_i \cdot \bar{\mathbf{q}}_i] \to \mathbb{R}$ for each bidder i which only depends on her activity points:*

$$c_i(a) = \mathbf{w}_i \cdot \mathbf{b}_i(a) \qquad\qquad 0 \leq a \leq \mathbf{r} \cdot \bar{\mathbf{q}}_i \tag{3.8}$$

Intuitively, for bidder i, $c_i(a)$ is the minimum cost of generating 'a' activity points.

At this point, we can define the bid vectors and all the equations only in terms of a_i. Bidders only need to specify their activity point a_i. We denote the final activity points of bidder i when the auction concluded by a_i^* and the final total activity point by A^*. The utility of each bidder i can now be written as the following:

$$u_i = v_i(\alpha^* a_i^*) - c_i(a_i^*) \tag{3.9}$$

Also, the auction concludes at the highest clock α^* such that:

$$\sum_{i=1}^{n} \alpha^* A^* = M \tag{3.10}$$

Next, we define the Nash-equilibrium. Before that, notice we can write the utility of each bidder i as $u_i(a, A)$ which is a function of her own bid and the total aggregate bid. Formally:

$$u_i(a, A) = v_i(\frac{a}{A} M) - c_i(a) \tag{3.11}$$

Now we are ready to describe the Nash-equilibrium. Suppose $a_1^*, \cdots, a_n^*$ are the activity points at which the auction has concluded. We say the outcome of the auction is *stable* or is a *Nash-equilibrium* if for every bidder i, a_i^* is a best response to a_{-i}^*. For a Nash equilibrium, the first order and boundary conditions are sufficient. Assume that $\bar{a}_i$ denotes the maximum possible activity points that bidder i can generate (i.e., $\bar{a}_i = \mathbf{r} \cdot \bar{q}_i$). The first order and boundary conditions of the Nash-equilibrium are the following:

$$\forall i \in N : \begin{cases} \frac{d}{da_i^*} u_i(a_i^*, A^*) = 0 \quad \text{and} \quad 0 < a_i^* < \bar{a}_i \\ \quad \text{or} \\ \frac{d}{da_i^*} u_i(a_i^*, A^*) \leq 0 \quad \text{and} \quad a_i^* = 0 \\ \quad \text{or} \\ \frac{d}{da_i^*} u_i(a_i^*, A^*) \geq 0 \quad \text{and} \quad a_i^* = \bar{a}_i \end{cases} \tag{3.12}$$

$$A^* = \sum_{i=1}^{n} a_i^* \tag{3.13}$$

Note that, to use the first order conditions, we need $u_i(a, A)$ to be a continuous and differentiable function in its domain. We can however relax the differentiability requirement and allow $u_i(a, A)$ to have different left and right derivatives at a finite number of points. In that case, if assume that ρ_i^- is the left derivative of $\frac{d}{da_i^*}u_i(a_i^*, A^*)$ and ρ_i^+ is its right derivative, then in the first condition, we can replace $\frac{d}{da_i^*}u_i(a_i^*, A^*) = 0$ with $\rho_i^- \leq 0 \leq \rho_i^+$. To keep the proofs simple, we do not use this general form but we will refer to it later when we explain how to compute the equilibrium.

We further expand the first order and boundary conditions. Notice that $\frac{d}{da_i^*}u_i(a_i^*, A^*) = \frac{\partial}{\partial a}u_i(a_i^*, A^*) + \frac{\partial}{\partial A}u_i(a_i^*, A^*)\frac{d}{da_i^*}A^*$. Because we always have $\frac{d}{da_i^*}A^* = 1$, we can write $\frac{d}{da_i^*}u_i(a_i^*, A^*) = \frac{\partial}{\partial a}u_i(a_i^*, A^*) + \frac{\partial}{\partial A}u_i(a_i^*, A^*)$. So the first order and boundary conditions can be rephrased as:

$$\forall i \in N : \begin{cases} \frac{\partial}{\partial a}u_i(a_i^*, A^*) + \frac{\partial}{\partial A}u_i(a_i^*, A^*) = 0 \quad \text{and} \quad 0 \leq a_i^* \leq \bar{a}_i \\ \quad\quad \text{or} \\ \frac{\partial}{\partial a}u_i(a_i^*, A^*) + \frac{\partial}{\partial A}u_i(a_i^*, A^*) \leq 0 \quad \text{and} \quad a_i^* = 0 \\ \quad\quad \text{or} \\ \frac{\partial}{\partial a}u_i(a_i^*, A^*) + \frac{\partial}{\partial A}u_i(a_i^*, A^*) \geq 0 \quad \text{and} \quad a_i^* = \bar{a}_i \end{cases} \tag{3.14}$$

$$A^* = \sum_{i=1}^{n} a_i^* \tag{3.15}$$

Next, we state the main theorem of this section which gives a sufficient condition for the existence of a Nash-equilibrium and provides a method for computing it as well as a bidding strategy.

Theorem 1. *Consider the Auction. 1, in which as explained before, each bidder's utility is given by* $u_i = v_i(\alpha\mathbf{r} \cdot \mathbf{b}_i) - \mathbf{c}_i \cdot \mathbf{b}_i$ *if the auction stops at the current clock* α*. Assuming that the valuation functions* v_i *are continuous, differentiable*[2] *and concave, there exists a unique Nash-equilibrium that satisfies the first order and boundary conditions of* (3.14)*. Furthermore, there are bid functions* $g_i(\alpha)$*, such that for every* i *if bidder* i *bids* $\mathbf{b}_i^*(a_i)$ *where* $a_i = g_i(\alpha)$*, then the outcome of the auction coincides with the unique Nash-equilibrium.* $g_i(\alpha)$ *is given bellow (*v_i' *and* c_i' *are the derivatives of* v_i *and* c_i*):*

$$g_i(\alpha) = \text{argmin}_{a \in [0, \bar{a}_i]} \left| v_i'(\alpha a)\frac{M - \alpha a}{M}\alpha - c_i'(a) \right| \tag{3.16}$$

Furthermore, the $g_i(\alpha)$ *can be computed efficiently using binary search on 'a' (the parameter of the* argmin*) because the expression inside the absolute value is a decreasing function of* a*.*

Note that the requirement of v_i functions being concave is quite natural. It simply means that the derivative of v_i should be decreasing which can be interpreted as the marginal value of the first dollar received being more than the marginal value of the last dollar.

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[2] we may relax this to allow v_i to have different left and right derivatives at a finite number of points

It is worth mentioning that the bid function $g_i(\alpha)$, as described in (3.16) is not necessarily an increasing function of α. In other words, as the clock goes down, $g_i(\alpha)$ may increase at some points which means bidder i is actually offering more for sale although the prices are going down. This phenomenon is in fact quite common when bidder i has liquidity needs as we will explain in section 5 .

We defer the proof of Theorem 1 to section 5. Instead of proving Theorem 1 directly, we prove a more general theorem in the next section. Later, in section 5, we show that Theorem 1 is a special case of that.

4 Summation Games

In this section, we describe a general class of games which we will refer to as summation games. Later, we show that the reverse auction explained in the previous section and some well known problems like the Courant-Nash equilibrium of an oligopoly game [7] can be expressed in this model. Next, we define a *Summation Game*:

Definition 2 (Summation Game). *There are n players $N = \{1, \cdots, n\}$. Each player can choose a number a_i from the interval $[0, \bar{a}_i]$ where $\bar{a}_i$ is a constant. The utility of each bidder depends only on her own number as well as the sum of all the numbers. In other words, assuming that $A = \sum_{i=1}^{n} a_i$, the utility of each bidder i is given by $u_i(a_i, A)$.*

We next show that if the utility functions $u_i(a, A)$ meet a certain requirement, the summation game has a unique Nash-equilibrium that can be computed efficiently. Before that, we define the following notation

Definition 3. *For each player i, assuming that $u_i(a, A)$ is her utility function, define her characteristic function $h_i(x, T)$ as the following:*

$$h_i(x, T) = \frac{\partial}{\partial a} u_i(xT, T) + \frac{\partial}{\partial A} u_i(xT, T) \tag{4.1}$$

Theorem 2. *If all the characteristic functions $h_i(x, T)$[3] are* strictly decreasing *functions in both x and T, then the game has a unique Nash-equilibrium [4] and in that equilibrium, the bid of each player i is $a_i = x_i(A)A$ where x_i is defined as the following:*

$$x_i(T) = \text{argmin}_{x \in [0, \min(1, \frac{\bar{a}_i}{T})]} |h_i(x, T)| \tag{4.2}$$

[3] Note that we allow $h_i(x, T)$ to be discontinuous at a finite number of points (e.g. a step function).

[4] if we relax the requirement of h_i's being strictly decreasing to just being *non-increasing* then there is a continuum of Nash-equilibria in which there is one Nash-equilibrium that is strictly preferred by some players and is just as good as other Nash equilibria for other players.

Furthermore, because $h_i(x,T)$ is decreasing in both x and T, $x_i(T)$ is also decreasing in T and the equilibrium can be computed efficiently using two nested binary searches or using an auction-like mechanism with an ascending clock T in which the each bidder i submits $a_i = x_i(T)T$ and the clock T keeps going up as long as $\sum_i a_i > T$.

Proof. First, it is easy to see that the first order and boundary conditions that are necessary and sufficient for the Nash-equilibrium are exactly those of (3.14) which we wrote for the Nash-equilibrium of Auction. 1. We can rewrite those conditions in terms of $h_i(x,T) = \frac{\partial}{\partial a}u_i(xT,T) + \frac{\partial}{\partial A}u_i(xT,T)$ for each player i as the following. Again, note that the following is just a restatement of (3.14) in which $x_i^* = \frac{a_i^*}{A^*}$ and $T^* = A^*$ and a_i^*'s are the equilibrium bids:

$$\forall i \in N : \begin{cases} h_i(x_i^*, T^*) = 0 \quad \text{and} \quad 0 \le x_i^* \le \min(1, \frac{\bar{a}_i}{T^*}) \\ \qquad \text{or} \\ h_i(x_i^*, T^*) \le 0 \quad \text{and} \quad x_i^* = 0 \\ \qquad \text{or} \\ h_i(x_i^*, T^*) \ge 0 \quad \text{and} \quad x_i^* = \min(1, \frac{\bar{a}_i}{T^*}) \end{cases} \tag{4.3}$$

$$\sum_{i=1}^{n} x_i^* = 1 \tag{4.4}$$

Based on our assumption that each $h_i(x,T)$ is a decreasing function of both x and T, it is easy to see that the above 3 conditions can be written in a compact form as the following single condition:

$$\forall i \in N : \quad x_i^* = \text{argmin}_{x_i^* \in [0, \min(1, \frac{\bar{a}_i}{T^*})]} |h_i(x_i^*, T^*)| \tag{4.5}$$

$$\sum_{i=1}^{n} x_i^* = 1 \tag{4.6}$$

To get an intuition of why (4.5) is equivalent to (4.3). Suppose for a given T^*, we want to find x_i^* that satisfies (4.5). Take any arbitrary $x \in [0, \min(1, \frac{\bar{a}_i}{T^*})]$. Since $h_i(x,T^*)$ is decreasing in x, to minimize $|h_i(x,T^*)|$, if $h_i(x,T^*)$ is negative we should decrease x until either $h_i(x,T^*)$ becomes 0; or x reaches 0 and $h_i(x,T^*)$ is still positive. Otherwise, if $h_i(x,T^*)$ is positive we would do the opposite. Note that $x_i(T)$ as defined in (4.2) returns the value of x that minimizes $|h_i(x,T)|$. Based on what we just explained, it is easy to see that for any given T, we can actually do a binary search to find the x that minimizes $|h_i(x,T)|$ and therefore we can efficiently compute $x_i(T)$ using a binary search even for fairly complex h_i. Next we show an important property of $x_i(T)$.

Lemma 1. *$x_i(T)$ is is a strictly decreasing*[5] *function of T.*

[5] When $h_i(x,T)$ functions are non-increasing in x and T instead if strictly decreasing then $x_i(T)$ may also be non-increasing instead of strictly decreasing

Proof. We only give a sketch of the proof. Take any given T, we know that $x_i(T)$ gives the x that makes $h_i(x, T)$ as close to 0 as possible. If we increase T by $\Delta T > 0$, that may only decrease $h_i(x, T)$ by some $\epsilon > 0$, if $h_i(x, T)$ was 0, now it becomes negative so to counter that and bring $h_i(x, T)$ close to 0 we have to decrease x by some $\Delta x > 0$. On the other hand, if $h_i(x, T)$ was positive[6] it means we are already in the case of $x = \min(1, \frac{\bar{a}_i}{T})$ which may actually cause x to decrease if $\frac{\bar{a}_i}{T} < 1$ because $\frac{\bar{a}_i}{T}$ decreases as T increases.

Finally, to find the values of a_i's at the Nash-equilibrium we can use the following algorithm:

Algorithm 2

- *Start with $T = 0$ (or a sufficiently small positive T).*
- *Keep increasing T for as long as $\sum_{i=1}^{n} x_i(T) > 1$.*
- *Stop as soon as $\sum_{i=1}^{n} x_i(T) \leq 1$ and then set each bid $a_i = x_i(T)T$* [7]

To see why the above algorithm works, we use Lemma 1 to argue that the value of the equilibrium aggregate bid [8] T^* and all the $x_i(T^*)$ values are unique. Because for any $T' > T^*$, $\sum_{i=1}^{n} x_i(T') < 1$ and for any $T' < T^*$, $\sum_{i=1}^{n} x_i(T') > 1$.

Note that Alg. 2 can be implemented either using binary search on T or as an ascending auction-like mechanism in which each player submits the bid $a_i = x_i(T)T$ where T is the ascending clock and in which the clock stops once $\sum_i a_i \leq T$.

In the next section we finish our analysis of the pooled reverse auction of 1. Later, in section 6, we give example of a well-known problem that can be expressed in our model and its Nash-equilibrium can be computed using Alg. 2.

5 Back to Pooled Reverse Auction

In the previous section we described a more general class of games and in Theorem 2 we gave sufficient conditions for the existence of a Nash-equilibrium. We explained when it is unique and how to compute it. In this section we continue our analysis of Auction. 1. We first give a proof for Theorem 1 be reducing it to a special case of Theorem 2.

Next, we give a proof for Theorem 1 which is based on a reduction to Theorem 2.

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[6] In case h_i is discontinuous at x and T, the proof will be slightly different

[7] If $\sum_{i=1}^{n} x_i(T^*) < 1$ then arbitrarily choose each a_i^* from the interval $[\lim_{\epsilon \to 0+} x_i(T^* - \epsilon)T^*, x_i(T^*)T^*]$ such that $\sum_{i=1}^{n} a_i^* = T^*$ (It is easy to show that each player i is indifferent to all $a_i^* \in [\lim_{\epsilon \to 0+} x_i(T^* - \epsilon)T^*, x_i(T^*)T^*]$).

[8] When $h_i(x, T)$ functions are non-increasing instead if strictly decreasing then this algorithm finds the equilibrium with the smallest aggregate bid T^*.

Proof (***Proof of. Theorem 1***). To be able to apply Theorem 2, we first need to show that the utility function of each bidder in Auction. 1 meets the requirement of Theorem 2. More specifically, we have to show that $h_i(x,T) = \frac{\partial}{\partial a}u_i(xT,T) + \frac{\partial}{\partial A}u_i(xT,T)$ is a decreasing function in both x and T. Remember that in our model for Auction. 1, we can write the utility of bidder i as $u_i(a,A) = v_i(\frac{a}{A}M) - c_i(a)$. First, we show that $c_i(a)$ is a convex function.

Lemma 1. *The cost function $c_i(x)$ as defined in (3.8) is always a convex function and has an non-decreasing first order derivative in $[0, \mathbf{r}.\bar{\mathbf{q}}]$ although its derivative might be discontinuous in at most m points.*

Proof. The proof is based on the construction given in the proof of Lemma 2. Note that based on the definition of $\mathbf{b}_i(a)$ from (3.6) and (3.7) $c_i(a)$ is a piecewise linear function consisting of m segments and its derivative is given by the following:

$$\frac{\partial}{\partial x}c_i(a) = \frac{w_y}{r_y} \quad : \sum_{j=1}^{y-1} r_j \bar{q}_{i,j} < a < \sum_{j=1}^{y} r_j \bar{q}_{i,j} \tag{5.1}$$

Since we assumed that securities are sorted such that $\frac{r_j}{w_j} \geq \frac{r_{j+1}}{w_{j+1}}$ we can see that the derivative of each segment of c_i is greater than or equal to the previous segment which means its derivative is non-decreasing and so c_i is convex.

Lemma 2. *The $h_i(x,T)$ functions for bidders in Auction. 1 are decreasing in both x and T:*

$$h_i(x,T) = \frac{\partial}{\partial a}u_i(xT,T) + \frac{\partial}{\partial A}u_i(xT,T) \tag{5.2}$$

$$= v_i'(xM)\frac{1-x}{T} + c_i'(Tx) \tag{5.3}$$

Proof. We showed in Lemma 2 that c_i is a convex function. We also assumed in Theorem 1 that v_i is a concave function. It is then easy to verify that (5.3) is indeed a decreasing function in both x and T. First, because v_i is concave v_i' is non-increasing, $\frac{1-x}{T}$ is decreasing in both x and T, c_i is convex and c_i' is non-decreasing which means $-c_i'(Tx)$ is non-increasing in both x and T. Putting them all together, $h_i(x,T)$ is a strictly decreasing function of both x and T.

Since in Lemma 2 we proved that $h_i(x,T)$ is decreasing in both x and T, we can now apply Theorem 2 and all of the claims of Theorem 1 follow from Theorem 2. Also note that $g_i(\alpha)$ which was defined in (3.16) is actually the same as $x_i(T)T$, where $T = \frac{M}{\alpha}$

It is interesting to notice that the auction-like mechanism of Theorem 2 and Auction. 1 are actually equivalent. In fact, the $x_i(T)$ where $T = M/\alpha$, has a very natural interpretation in Auction. 1. It specifies the fraction of the budget of the auctioneer that the bidder i is demanding at the clock α. In fact we may modify the auction to ask the bidders to submit the amount of liquidity that they are demanding directly at each step of the clock and then the auction stops when the demand becomes less than or equal to the budget of the auctioneer. Then, each bidder will be required to sell enough quantity of her shares at the current prices to pay for the liquidity that she had demanded.

It is easy to see that the liquidity that each bidder demands may only decrease as the α increases. However, the value of the bid, $x_i(T)T$, may actually increase because bidder I may want to maintain her demand for the liquidity.

6 Application to Cournot's Oligopoly

In this section, we show how the well-known problem of *Cournot's Oligopoly* can be expressed in our model of a summation game and all the results of Theorem 2 can therefore be applied:

Definition 4 (Cournot's Oligopoly).

- *There are n firms. The firms are oligopolist suppliers of a homogenous good.*
- *At each period, each firm chooses a quantity q_i to supply.*
- *The total supply Q on the market is the sum of all firms' supplies:*

$$Q = \sum_i q_i \tag{6.1}$$

- *All firms receive the same price p per unit of the good. The price p on the market depends on the total supply Q as:*

$$p(Q) = p_0(Q_{\max} - Q) \tag{6.2}$$

- *Each firm i incurs a cost c_i per unit of good. These costs can be different for different firms and are private information*
- *Each firm i's profit is given by:*

$$u_i(q_i, Q) = (p(Q) - c_i)q_i \tag{6.3}$$

- *After each market period, firms are informed of the total quantity Q and the market price $p(Q)$ of the previous period.*

If we write down the $h_i(x, T)$ for each firm i we get:

$$h_i(x, T) = \frac{\partial}{\partial a}u_i(xT, T) + \frac{\partial}{\partial A}u_i(xT, T) \tag{6.4}$$

$$= p(T) - c_i + p'(T)Tx \tag{6.5}$$

$$= p_0(Q_{\max} - T) - c_i - p_0 T x \tag{6.6}$$

Notice that clearly the above $h_i(x, T)$ is a decreasing function of both x and T and therefore all of the nice results of Theorem 2 can be applied. Notice in fact that as long as $p(Q)$ is concave and a decreasing function of Q, $h_i(x, T)$ is still a decreasing function of both x and T and all of the results of Theorem 2 still holds.

7 Conclusion

In this paper we studied the Nash-equilibrium and equilibrium bidding strategies of the troubled assets reverse auction. We further generalized our analysis to a more general class of games with non quasi-linear utilities. We proved the existence and uniqueness of a Nash-equilibrium in those games and we also gave an efficient way to compute the equilibrium of those games. We also showed that finding the Nash equilibrium can be implemented using an ascending mechanism so that the participants don't need to reveal their utility functions. We also, showed that even a more general version of the well-known problem of Cournot's Oligopoly can be expressed in our model and all of the previously mentioned results apply to that as well.

References

1. L. Ausubel and P. Cramton. A troubled asset reverse auction. In *Working paper*, 2008.
2. L. M. Ausubel, P. Cramton, E. Filiz-Ozbay, N. Higgins, E. Ozbay, and A. Stocking. Common-value auctions with liquidity needs:an experimental test of a troubled assets reverse auction. University of maryland, economics working paper series, Department of Economics, University of Maryland, 2008.
3. T. Bergstrom and H. Varian. Two remarks on cournot equilibria. University of california at santa barbara, economics working paper series, Department of Economics, UC Santa Barbara, Feb. 1985.
4. R. Johari, S. Mannor, and J. N. Tsitsiklis. Efficiency loss in a network resource allocation game. *Mathematics of Operations Research*, 29:407–435, 2004.
5. R. Johari and J. Tsitsiklis. Efficiency of scalar-parameterized mechanisms. *Operation Research*, 2008.
6. F. Kelly. Charging and rate control for elastic traffic. *European Transactions on Telecommunications*, 8:33–37, 1997.
7. A. Mas-colell, M. D. Whinston, and G. J. R.
8. S. Shenker, D. Clark, D. Estrin, and S. Herzog. Pricing in computer networks: Reshaping the research agenda. *ACM Computer Communication Review*, 26:183–201, 1996.