Simplified Reducibility Proofs of Church-Rosser for β- and $\beta\eta$-reduction

Fairouz Kamareddine and Vincent Rahli

ULTRA group, MACS, Heriot Watt University

26 August 2008

Interest

We prove the Church-Rosser property of the untyped λ-calculus w.r.t. β- and $\beta\eta$-reductions.

- Simplification and generalisation of some semantic proofs of the Church-Rosser property.
  (Based on type interpretations w.r.t. a given type system.)
- Only a small portion of the type systems considered are actually needed.
- This portion corresponds to a few simple sets of terms satisfying simple closure properties.
- We obtain a syntactic proof projectable in a semantic framework.

Steps to prove the Church-Rosser property

Usual steps of a proof of the Church-Rosser property of a λ-calculus:

- Definition of the developments.
- Proof of the confluence of the developments.
- Equivalence between:
  - the transitive closure of the developments
  - the reflexive and transitive closure of the reduction relation of the considered calculus (for example the β-reduction).

The proofs of the Church-Rosser property can be divided as follows:

- First division:
  - Encoding the development using a reduction relation: Tait and Martin-Löf [Lév76], Takahashi [Tak89].
  - Encoding the development using a set of terms: Barendregt et al. [BBKV76], Ghilezan and Kunčak [GK01], Koletsos and Stavrinos [KS07].
- Second division:
  - Using a semantic method: Koletsos and Stavrinos [KS07].
  - Using a syntactic method: Barendregt et al. [BBKV76], Tait and Martin-Löf [Lév76], Takahashi [Tak89].

The Church-Rosser property - our contribution

We simplified and extended the semantic proof of Koletsos and Stavrinos [KS07] to obtain a syntactic proof.

- Our proof is based on the encoding of developments using a set of terms rather than a reduction relation.
- We do not deal with types as Ghilezan and Kunčak [GK01] or Koletsos and Stavrinos [KS07].
- Our proof is simpler than other similar syntactic proofs such as the one of Barendregt et al. [BBKV76].
- Our proof of the confluence of developments is parametric (we can easily prove the finiteness of developments).
- Our proof can be seen as a bridge between semantic proofs (e.g., by Koletsos and Stavrinos) and syntactic proofs (e.g., by Barendregt et al).

The Church-Rosser property - our contribution
The needed machinery - the set of terms

The λ-calculus is built on the set of terms Λ and β-reduction:

$$M \in \Lambda ::= x \mid \lambda x.M \mid M_1 M_2 \qquad \text{where } x \in \text{Var}$$

Our **developments** are based on the following parametric sets of terms:
$(\Lambda_c^{\beta} \subset \Lambda_c^{\beta\eta} \subset \Lambda)$
For the β-case (Krivine [Kri90]):

$$\bar{M} \in \Lambda_c^{\beta} ::= \bar{x} \mid \lambda \bar{x}.\bar{M} \mid c\bar{M}_1\bar{M}_2 \mid (\lambda \bar{x}.\bar{M}_1)\bar{M}_2$$

For the $\beta\eta$-case:

$$\bar{M} \in \Lambda_c^{\beta\eta} ::= \bar{x} \mid \lambda \bar{x}.\bar{M} \mid c\bar{M}_1\bar{M}_2 \mid (\lambda \bar{x}.\bar{M}_1)\bar{M}_2 \mid c\bar{M}$$

where $\bar{x} \in \text{Var} \setminus \{c\}$

- A freezing function:
  - $\Psi_c(x) = \Psi_c(x)$
  - $\Psi_c(\lambda x.M) = \lambda x.\Psi_c(M)$
  - $\Psi_c(M_1 M_2) = \Psi_c(M_1)\Psi_c(M_2)$ if M_1 is a λ-abstraction
  - $\Psi_c(M_1 M_2) = c\Psi_c(M_1)\Psi_c(M_2)$ otherwise

  The freezing function freezes the "potential" β-redexes of a term (it does not freeze the η-redexes).
- An erasure relation based on:

$$cM \rightarrow_c M$$

  This relation enables to unfreeze a frozen term.

Let $M = (\lambda x.xx)(\lambda x.yx)$.

$$\Psi_c(M) = (\lambda x.cxx)(\lambda x.cyx)$$

$\Psi_c(M)$ can $\beta\eta$-reduce as follows:

$$(\lambda x.cxx)(\lambda x.cyx) \to_\eta (\lambda x.cxx)(cy) \to_\beta c(cy)(cy) = P$$

M can $\beta\eta$-reduce as follows:

$$(\lambda x.xx)(\lambda x.yx) \to_\eta (\lambda x.xx)y \to_\beta yy = Q$$

We erase the c's from P to obtain Q.

$$c(cy)(cy) \to_c cy(cy) \to_c y(cy) \to_c yy$$

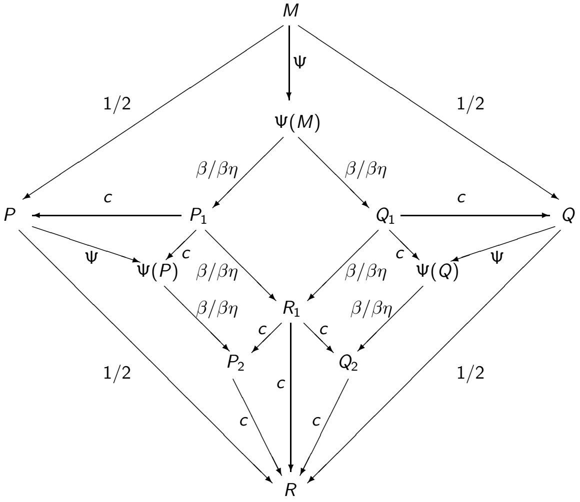
The β-case:

$$M \rightarrow_1 N \iff \Psi_c(M) \rightarrow_\beta^* P \rightarrow_c^* N \wedge c \notin \mathrm{fv}(M) \cup \mathrm{fv}(N)$$

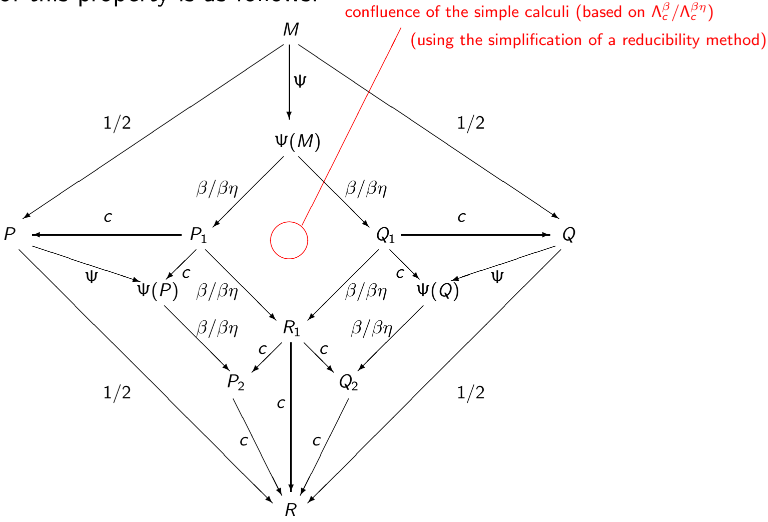
The $\beta\eta$-case:

$$M \rightarrow_2 N \iff \Psi_c(M) \rightarrow_{\beta\eta}^* P \rightarrow_c^* N \wedge c \notin \mathrm{fv}(M) \cup \mathrm{fv}(N)$$

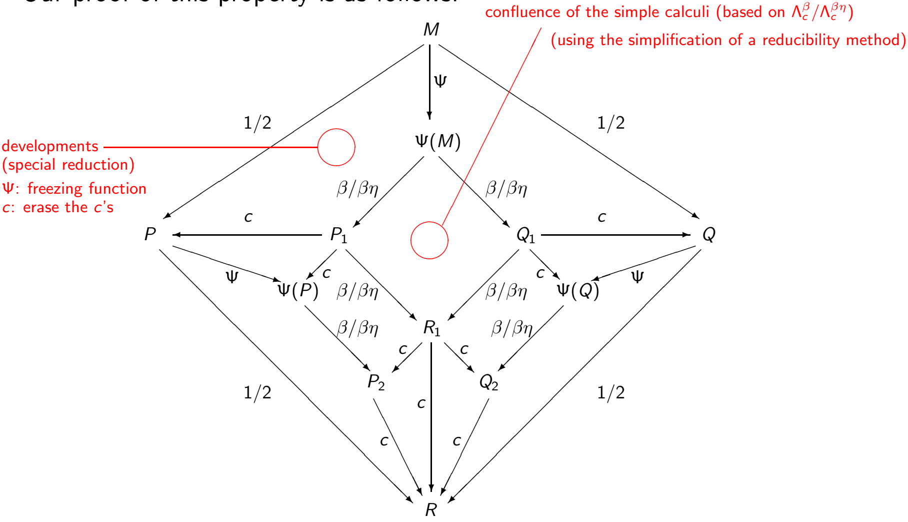
Our proof of this property is as follows:



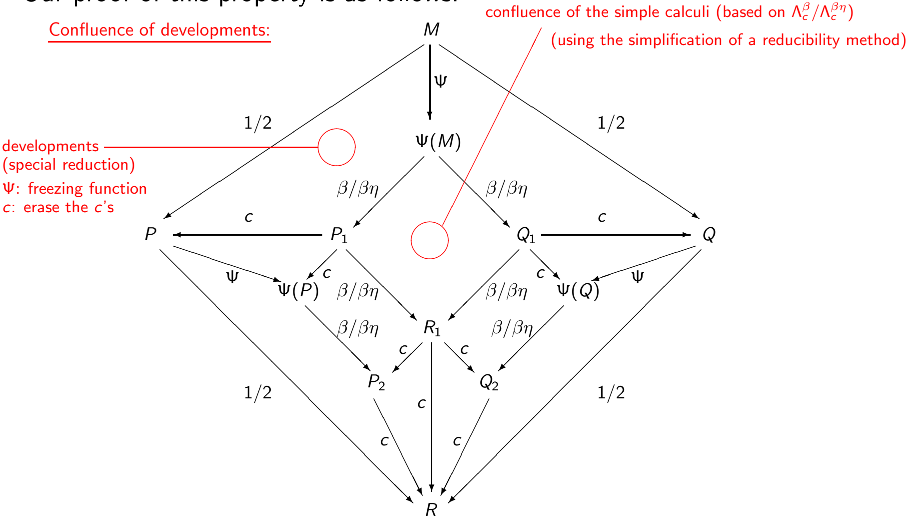
Our proof of this property is as follows:



Our proof of this property is as follows:

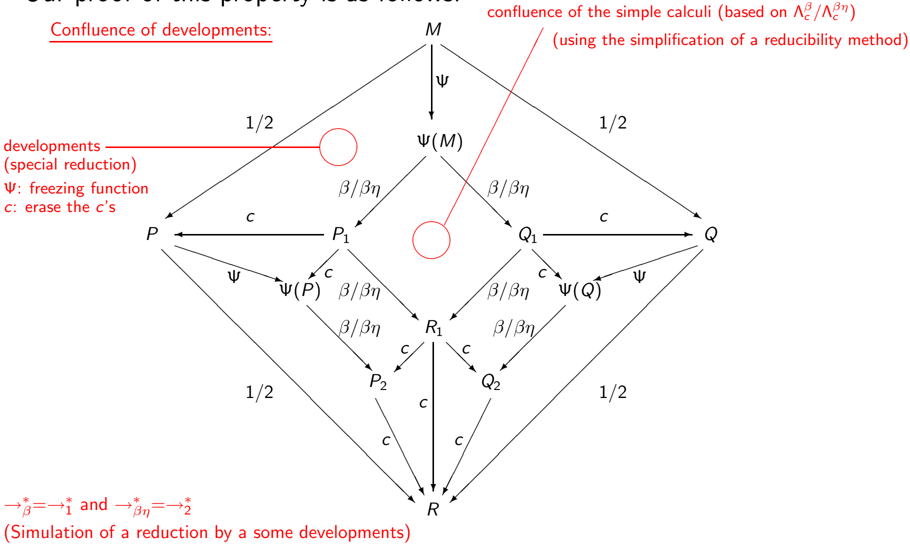


Our proof of this property is as follows:



The Church-Rosser property - our contribution
The method

Our proof of this property is as follows:



Comparison with other proofs

We believe our β-developments to be equivalent to those of Church and Rosser [CR36], Barendregt et al. [BBKV76], Ghilezan and Kunčak [GK01], Koletsos and Stavrinos [KS07].

- Barendregt et al. [BBKV76]: We do not introduce new terms; we do not need the completeness of developments.
- Ghilezan and Kunčak [GK01]: We do not need a type system; the embedding of developments is simpler.
- Koletsos and Stavrinos [KS07]: We do not need a type system; we do not deal with residuals.

The scheme of our proof method is similar to those cited above.

Comparison with other proofs

Our β-developments allow strictly more reductions than those of Tait and Martin-Löf [Lév76].

Let $M = (\lambda x.xx)((\lambda z.z)y)$. We have:

- $\Psi_c(M) = (\lambda x.cxx)((\lambda z.z)y) \rightarrow_\beta c((\lambda z.z)y))((\lambda z.z)y) \rightarrow_\beta$
  $cy((\lambda z.z)y) \rightarrow_c y((\lambda z.z)y)$
- $M \rightarrow_1 y((\lambda z.z)y)$
- $M \not\twoheadrightarrow_\beta y((\lambda z.z)y)$

This is because we allow different residuals of the same redex to reduce independently.

Comparison with other proofs

Our $\beta\eta$-developments allow strictly more reductions than those of Takahashi [Tak89].

Let $M = \lambda x.y((\lambda z.z)x)$. We have:

- $\Psi_c(M) = \lambda x.cy((\lambda z.z)x) \rightarrow_\beta \lambda x.cyx \rightarrow_\eta cy \rightarrow_c y$
- $M = \lambda x.y((\lambda z.z)x) \rightarrow_2 y$
- $M \Rightarrow_{\beta\eta} y$

This is because we allow the reduction of any η-redex even if it is not the residual of an η-redex.

Comparison with other proofs

Hence:

- The obtained bridge between syntactic and semantic methods is (for example) between Barendregt et al. [BBKV76] and Koletsos and Stavrinos [KS07].
- It is not between Takahashi [Tak89] and Koletsos and Stavrinos [KS07].

H. Barendregt, J. A. Bergstra, J. W. Klop, and H. Volken.
Degrees, reductions and representability in the lambda calculus.
Technical Report Preprint no. 22, University of Utrecht, Department of Mathematics, 1976.

Alonzo Church and John B. Rosser.
Some properties of conversion.
Transactions of the American Mathematical Society, 39(3):472–482, 1936.

Silvia Ghilezan and Viktor Kunčak.
Confluence of untyped lambda calculus via simple types.
Lecture Notes in Computer Science, 2202:38–49, 2001.

J. L. Krivine.
Lambda-calcul, types et modèles.
Dunod, 1990.

G. Koletsos and G. Stavrinos.
Church-Rosser property and intersection types.
Australasian Journal of Logic, 2007.

Jean-Jacques Lévy.
An algebraic interpretation of the *lambda beta* k-calculus; and an application of a labelled *lambda* -calculus.
Theoretical Compututer Science, 2(1):97–114, 1976.

Masako Takahashi.
Parallel reductions in lambda-calculus.
Journal of Symbolic Computation, 7(2):113–123, 1989.