A complete realisability semantics for intersection types and arbitrary expansion variables

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Abstract. *Expansion* was introduced at the end of the 1970s for calculating *principal typings* for λ-terms in intersection type systems. *Expansion variables* (E-variables) were introduced at the end of the 1990s to simplify and help mechanise expansion. Recently, E-variables have been further simplified and generalised to also allow calculating other type operators than just intersection. There has been much work on semantics for intersection type systems, but only one such work on intersection type systems with E-variables. That work established that building a semantics for E-variables is very challenging. Because it is unclear how to devise a space of meanings for E-variables, that work developed instead a space of meanings for types that is hierarchical in the sense of having many degrees (denoted by indexes). However, although the indexed calculus helped identify the serious problems of giving a semantics for expansion variables, the sound realisability semantics was only complete when one single E-variable is used and furthermore, the universal type ω was not allowed. In this paper, we are able to overcome these challenges. We develop a realisability semantics where we allow an arbitrary (possibly infinite) number of expansion variables and where ω is present. We show the soundness and completeness of our proposed semantics.

1 Introduction

Expansion is a crucial part of a procedure for calculating *principal typings* and thus helps support compositional type inference. For example, the λ-term $M = (\lambda x.x(\lambda y.yz))$ can be assigned the typing $\Phi_1 = \langle (z : a) \vdash (((a \to b) \to b) \to c) \to c \rangle$, which happens to be its principal typing. The term M can also be assigned the typing $\Phi_2 = \langle (z : a_1 \sqcap a_2) \vdash (((a_1 \to b_1) \to b_1) \sqcap ((a_2 \to b_2) \to b_2) \to c) \to c \rangle$, and an expansion operation can obtain Φ_2 from Φ_1. Because the early definitions of expansion were complicated [4], E-variables were introduced in order to make the calculations easier to mechanise and reason about. For example, in System E [2], the above typing Φ_1 is replaced by $\Phi_3 = \langle (z : ea) \vdash e((((a \to b) \to b) \to c) \to c) \rangle$, which differs from Φ_1 by the insertion of the E-variable e at two places, and Φ_2

can be obtained from Φ_3 by substituting for e the *expansion term*:
$E = (a := a_1, b := b_1) \sqcap (a := a_2, b := b_2)$.

Carlier and Wells [3] have surveyed the history of expansion and also E-variables. Kamareddine, Nour, Rahli and Wells [13] showed that E-variables pose serious challenges for semantics. In the list of open problems published in 1975 in [6], it is suggested that an arrow type expresses functionality. Following this idea, a type's semantics is given as a set of closed λ-terms with behaviour related to the specification given by the type. In many kinds of semantics, the meaning of a type T is calculated by an expression $[T]_\nu$ that takes two parameters, the type T and a valuation ν that assigns to type variables the same kind of meanings that are assigned to types. In that way, models based on term-models have been built for intersection type systems [7, 14, 11] where intersection types (introduced to type more terms than in the Simply Typed Lambda Calculus) are interpreted by set-theoretical intersection of meanings. To extend this idea to types with E-variables, we need to devise some space of possible meanings for E-variables. Given that a type $e\,T$ can be turned by expansion into a new type $S_1(T) \sqcap S_2(T)$, where S_1 and S_2 are arbitrary substitutions (or even arbitrary further expansions), and that this can introduce an unbounded number of new variables (both E-variables and regular type variables), the situation is complicated.

This was the main motivation for [13] to develop a space of meanings for types that is hierarchical in the sense of having many degrees. When assigning meanings to types, [13] captured accurately the intuition behind E-variables by ensuring that each use of E-variables simply changes degrees and that each E-variable acts as a kind of capsule that isolates parts of the λ-term being analysed by the typing.

The semantic approach used in [13] is realisability semantics along the lines in Coquand [5] and Kamareddine and Nour [11]. Realisability allows showing *soundness* in the sense that the meaning of a type T contains all closed λ-terms that can be assigned T as their result type. This has been shown useful in previous work for characterising the behaviour of typed λ-terms [14]. One also wants to show the converse of soundness which is called *completeness* (see Hindley [8–10]), i.e., that every closed λ-term in the meaning of T can be assigned T as its result type. Moreover, [13] showed that if more than one E-variable is used, the semantics is not complete. Furthermore, the degrees used in [13] made it difficult to allow the universal type ω and this limited the study to the λI-calculus. In this paper, we are able to overcome these challenges. We develop a realisability semantics where we allow the full λ-calculus, an arbitrary (possibly infinite) number of expansion variables and where ω is present, and we show its soundness and completeness. We do so by introducing an indexed calculus as in [13]. However here, our indices are finite sequences of natural numbers rather than single natural numbers.

In Section 2 we give the full λ-calculus indexed with finite sequences of natural numbers and show the confluence of β, $\beta\eta$ and weak head reduction on the indexed λ-calculus. In Section 3 we introduce the type system for the indexed λ-calculus (with the universal type ω). In this system, intersections and expansions cannot occur directly to the right of an arrow. In Section 4 we establish that subject reduction holds for $\vdash$. In Section 5 we show that subject β-expansion holds for $\vdash$ but that subject η-expansion fails. In Section 6 we introduce the realisability semantics and show its soundness for $\vdash$. In Section 7 we establish the completeness of $\vdash$ by introducing a special interpretation. We conclude in Section 8. Omitted proofs can be found in the appendix.

2 The pure $\lambda^{\mathcal{L}_{\mathbb{N}}}$-calculus

In this section we give the λ-calculus indexed with finite sequences of natural numbers and show the confluence of β, $\beta\eta$ and weak head reduction.

Let n, m, i, j, k, l be metavariables which range over the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. We assume that if a metavariable v ranges over a set s then v_i and v', v'', etc. also range over s. A binary relation is a set of pairs. Let *rel* range over binary relations. We sometimes write $x\ rel\ y$ instead of $\langle x, y\rangle \in rel$. Let $\mathrm{dom}(rel) = \{x \mathrel{/} \langle x, y\rangle \in rel\}$ and $\mathrm{ran}(rel) = \{y \mathrel{/} \langle x, y\rangle \in rel\}$. A function is a binary relation *fun* such that if $\{\langle x, y\rangle, \langle x, z\rangle\} \subseteq fun$ then $y = z$. Let *fun* range over functions. Let $s \to s' = \{fun \mathrel{/} \mathrm{dom}(fun) \subseteq s \wedge \mathrm{ran}(fun) \subseteq s'\}$. We sometimes write $x : s$ instead of $x \in s$.

First, we introduce the set $\mathcal{L}_{\mathbb{N}}$ of indexes with an order relation on indexes.

Definition 1. *1. An index is a finite sequence of natural numbers $L = (n_i)_{1 \le i \le l}$. We denote $\mathcal{L}_{\mathbb{N}}$ the set of indexes and $\oslash$ the empty sequence of natural numbers. We let L, K, R range over $\mathcal{L}_{\mathbb{N}}$.*

2. If $L = (n_i)_{1 \le i \le l}$ and $m \in \mathbb{N}$, we use $m :: L$ to denote the sequence $(r_i)_{1 \le i \le l+1}$ where $r_1 = m$ and for all $i \in \{2, \ldots, l+1\}$, $r_i = n_{i-1}$. In particular, $k :: \oslash = (k)$.

3. If $L = (n_i)_{1 \le i \le n}$ and $K = (m_i)_{1 \le i \le m}$, we use $L :: K$ to denote the sequence $(r_i)_{1 \le i \le n+m}$ where for all $i \in \{1, \ldots, n\}$, $r_i = n_i$ and for all $i \in \{n+1, \ldots, n+m\}$, $r_i = m_{i-n}$. In particular, $L :: \oslash = \oslash :: L = L$.

4. We define on $\mathcal{L}_{\mathbb{N}}$ a binary relation $\preceq$ by: $L_1 \preceq L_2$ (or $L_2 \succeq L_1$) if there exists $L_3 \in \mathcal{L}_{\mathbb{N}}$ such that $L_2 = L_1 :: L_3$.

Lemma 1. *$\preceq$ is an order relation on $\mathcal{L}_{\mathbb{N}}$.*

The next definition gives the syntax of the indexed calculus and the notions of reduction.

Definition 2. *1. Let $\mathcal{V}$ be a countably infinite set of variables. The set of terms $\mathcal{M}$, the set of free variables $\mathrm{fv}(M)$ of a term $M \in \mathcal{M}$, the degree function $d : \mathcal{M} \to \mathcal{L}_{\mathbb{N}}$ and the joinability $M \diamond N$ of terms M and N are defined by simultaneous induction as follows:*

- *If $x \in \mathcal{V}$ and $L \in \mathcal{L}_{\mathbb{N}}$, then $x^L \in \mathcal{M}$, $\mathrm{fv}(x^L) = \{x^L\}$ and $d(x^L) = L$.*
- *If $M, N \in \mathcal{M}$, $d(M) \preceq d(N)$ and $M \diamond N$ (see below), then $M\ N \in \mathcal{M}$, $\mathrm{fv}(MN) = \mathrm{fv}(M) \cup \mathrm{fv}(N)$ and $d(M\ N) = d(M)$.*
- *If $x \in \mathcal{V}$, $M \in \mathcal{M}$ and $L \succeq d(M)$, then $\lambda x^L.M \in \mathcal{M}$, $\mathrm{fv}(\lambda x^L.M) = \mathrm{fv}(M) \setminus \{x^L\}$ and $d(\lambda x^L.M) = d(M)$.*

2. – *Let $M, N \in \mathcal{M}$. We say that M and N are joinable and write $M \diamond N$ iff for all $x \in \mathcal{V}$, if $x^L \in \text{fv}(M)$ and $x^K \in \text{fv}(N)$, then $L = K$.*
   – *If $\mathcal{X} \subseteq \mathcal{M}$ such that for all $M, N \in \mathcal{X}, M \diamond N$, we write, $\diamond\mathcal{X}$.*
   – *If $\mathcal{X} \subseteq \mathcal{M}$ and $M \in \mathcal{M}$ such that for all $N \in \mathcal{X}, M \diamond N$, we write, $M \diamond \mathcal{X}$.*

   The $\diamond$ property ensures that in any term M, variables have unique degrees. We assume the usual definition of subterms and the usual convention for parentheses and their omission (see Barendregt [1] and Krivine [14]). Note that every subterm of $M \in \mathcal{M}$ is also in $\mathcal{M}$. We let x, y, z, etc. range over $\mathcal{V}$ and M, N, P range over $\mathcal{M}$ and use $=$ for syntactic equality.
3. *The usual simultaneous substitution $M[(x_i^{L_i} := N_i)_n]$ of $N_i \in \mathcal{M}$ for all free occurrences of $x_i^{L_i}$ in $M \in \mathcal{M}$ is only defined when $\diamond\{M\} \cup \{N_i \ / \ i \in \{1, \ldots, n\}\}$ and for all $i \in \{1, \ldots, n\}$, $d(N_i) = L_i$. In a substitution, we sometimes write $x_1^{L_1} := N_1, \ldots, x_n^{L_n} := N_n$ instead of $(x_i^{L_i} := N_i)_n$. We sometimes write $M[(x_i^{L_i} := N_i)_1$ as $M[x_1^{L_1} := N_1]$.*
4. *We take terms modulo α-conversion given by:*
   $\lambda x^L.M = \lambda y^L.(M[x^L := y^L])$ *where for all L, $y^L \notin \text{fv}(M)$.*
   Moreover, we use the Barendregt convention (BC) where the names of bound variables differ from the free ones and where we rewrite terms so that not both λx^L and λx^K co-occur when $L \neq K$.
5. *A relation rel on $\mathcal{M}$ is compatible iff for all $M, N, P \in \mathcal{M}$:*
   – *If M rel N and $\lambda x^L.M, \lambda x^L.M \in \mathcal{M}$ then $(\lambda x^L.M)$ rel $(\lambda x^L.N)$.*
   – *If M rel N and $MP, NP \in \mathcal{M}$ (resp. $PM, PN \in \mathcal{M}$), then (MP) rel (NP) (resp. (PM) rel (PN)).*
6. *The reduction relation $\rhd_\beta$ on $\mathcal{M}$ is defined as the least compatible relation closed under the rule: $(\lambda x^L.M)N \rhd_\beta M[x^L := N]$ if $d(N) = L$*
7. *The reduction relation $\rhd_\eta$ on $\mathcal{M}$ is defined as the least compatible relation closed under the rule: $\lambda x^L.(M\ x^L) \rhd_\eta M$ if $x^L \notin \text{fv}(M)$*
8. *The weak head reduction $\rhd_h$ on $\mathcal{M}$ is defined by:*
   $(\lambda x^L.M)NN_1 \ldots N_n \rhd_h M[x^L := N]N_1 \ldots N_n$ *where $n \geq 0$*
9. *We let $\rhd_{\beta\eta} = \rhd_\beta \cup \rhd_\eta$. For $r \in \{\beta, \eta, h, \beta\eta\}$, we denote by $\rhd_r^*$ the reflexive and transitive closure of $\rhd_r$ and by $\simeq_r$ the equivalence relation induced by $\rhd_r^*$.*

The next theorem whose proof can be found in [12] states that free variables and degrees are preserved by our notions of reduction.

Theorem 1. *Let $M \in \mathcal{M}$ and $r \in \{\beta, \beta\eta, h\}$.*

1. *If $M \rhd_\eta^* N$ then $\text{fv}(N) = \text{fv}(M)$ and $d(M) = d(N)$.*
2. *If $M \rhd_r^* N$ then $\text{fv}(N) \subseteq \text{fv}(M)$ and $d(M) = d(N)$.*

As expansions change the degree of a term, indexes in a term need to increase/decrease.

Definition 3. *Let $i \in \mathbb{N}$ and $M \in \mathcal{M}$.*

1. *We define M^{+i} by:*
   $\bullet (x^L)^{+i} = x^{i::L}$ $\quad \bullet (M_1\ M_2)^{+i} = M_1^{+i}\ M_2^{+i}$ $\quad \bullet (\lambda x^L.M)^{+i} = \lambda x^{i::L}.M^{+i}$
   Let $M^{+\oslash} = M$ and $M^{+(i::L)} = (M^{+i})^{+L}$.

2. *If $d(M) = i :: L$, we define M^{-i} by:*
$\bullet (x^{i::K})^{-i} = x^K$ $\bullet (M_1\ M_2)^{-i} = M_1^{-i}\ M_2^{-i}$ $\bullet (\lambda x^{i::K}.M)^{-i} = \lambda x^K.M^{-i}$
Let $M^{-\oslash} = M$ and if $d(M) \succeq i :: L$ then $M^{-(i::L)} = (M^{-i})^{-L}$.
3. *Let $\mathcal{X} \subseteq \mathcal{M}$. We write $\mathcal{X}^{+i}$ for $\{M^{+i} \ / \ M \in \mathcal{X}\}$.*

Normal forms are defined as usual.

Definition 4. *1. $M \in \mathcal{M}$ is in β-normal form ($\beta\eta$-normal form, h-normal form resp.) if there is no $N \in \mathcal{M}$ such that $M \rhd_\beta N$ ($M \rhd_{\beta\eta} N$, $M \rhd_h N$ resp.).*
2. $M \in \mathcal{M}$ is β-normalising ($\beta\eta$-normalising, h-normalising resp.) if there is an $N \in \mathcal{M}$ such that $M \rhd_\beta^* N$ ($M \rhd_{\beta\eta} N$, $M \rhd_h N$ resp.) and N is in β-normal form ($\beta\eta$-normal form, h-normal form resp.).

The next theorem states that all of our notions of reduction are confluent on our indexed calculus. For a proof see [12].

Theorem 2 (Confluence). *Let $M, M_1, M_2 \in \mathcal{M}$ and $r \in \{\beta, \beta\eta, h\}$.*

1. *If $M \rhd_r^* M_1$ and $M \rhd_r^* M_2$, then there is M' such that $M_1 \rhd_r^* M'$ and $M_2 \rhd_r^* M'$.*
2. *$M_1 \simeq_r M_2$ iff there is a term M such that $M_1 \rhd_r^* M$ and $M_2 \rhd_r^* M$.*

3 Typing system

This paper studies a type system for the indexed λ-calculus with the universal type ω. In this type system, in order to get subject reduction and hence completeness, intersections and expansions cannot occur directly to the right of an arrow (see $\mathbb{U}$ below).

The next two definitions introduce the type system.

Definition 5. *1. Let a range over a countably infinite set $\mathcal{A}$ of atomic types and let e range over a countably infinite set $\mathcal{E} = \{\overline{e}_0, \overline{e}_1, ...\}$ of expansion variables. We define sets of types $\mathbb{T}$ and $\mathbb{U}$, such that $\mathbb{T} \subseteq \mathbb{U}$, and a function $d : \mathbb{U} \to \mathcal{L}_\mathbb{N}$ by:*
– *If $a \in \mathcal{A}$, then $a \in \mathbb{T}$ and $d(a) = \oslash$.*
– *If $U \in \mathbb{U}$ and $T \in \mathbb{T}$, then $U \to T \in \mathbb{T}$ and $d(U \to T) = \oslash$.*
– *If $L \in \mathcal{L}_\mathbb{N}$, then $\omega^L \in \mathbb{U}$ and $d(\omega^L) = L$.*
– *If $U_1, U_2 \in \mathbb{U}$ and $d(U_1) = d(U_2)$, then $U_1 \sqcap U_2 \in \mathbb{U}$ and $d(U_1 \sqcap U_2) = d(U_1) = d(U_2)$.*
– *$U \in \mathbb{U}$ and $\overline{e}_i \in \mathcal{E}$, then $\overline{e}_i U \in \mathbb{U}$ and $d(\overline{e}_i U) = i :: d(U)$.*
Note that d remembers the number of the expansion variables $\overline{e}_i$ in order to keep a trace of these variables.
We let T range over $\mathbb{T}$, and U, V, W range over $\mathbb{U}$. We quotient types by taking $\sqcap$ to be commutative (i.e. $U_1 \sqcap U_2 = U_2 \sqcap U_1$), associative (i.e. $U_1 \sqcap (U_2 \sqcap U_3) = (U_1 \sqcap U_2) \sqcap U_3$) and idempotent (i.e. $U \sqcap U = U$), by assuming the distributivity of expansion variables over $\sqcap$ (i.e. $e(U_1 \sqcap U_2) = eU_1 \sqcap eU_2$) and by having ω^L as a neutral (i.e. $\omega^L \sqcap U = U$). We denote $U_n \sqcap U_{n+1} \ldots \sqcap U_m$ by $\sqcap_{i=n}^m U_i$ (when $n \leq m$). We also assume that for all $i \geq 0$ and $K \in \mathcal{L}_\mathbb{N}$, $\overline{e}_i \omega^K = \omega^{i::K}$.

2. *We denote* $\overline{e}_{i_1}\dots\overline{e}_{i_n}$ *by* $\boldsymbol{e}_K$, *where* $K=(i_1,\dots,i_n)$ *and* $U_n\sqcap U_{n+1}\dots\sqcap U_m$ *by* $\sqcap_{i=n}^m U_i$ *(when* $n\le m$*).*

Definition 6. 1. *A type environment is a set* $\{x_1^{L_1}:U_1,\dots,x_n^{L_n}:U_n\}$ *such that for all* $i,j\in\{1,\dots,n\}$, *if* $x_i^{L_i}=x_j^{L_j}$ *then* $U_i=U_j\}$. *We let Env be the set of environments, use* Γ,Δ *to range over Env and write* $()$ *for the empty environment. We define* $\mathrm{dom}(\Gamma)=\{x^L\ /\ x^L:U\in\Gamma\}$. *If* $\mathrm{dom}(\Gamma_1)\cap\mathrm{dom}(\Gamma_2)=\emptyset$, *we write* Γ_1,Γ_2 *for* $\Gamma_1\cup\Gamma_2$. *We write* $\Gamma,x^L:U$ *for* $\Gamma,\{x^L:U\}$ *and* $x^L:U$ *for* $\{x^L:U\}$. *We denote* $x_1^{L_1}:U_1,\dots,x_n^{L_n}:U_n$ *by* $(x_i^{L_i}:U_i)_n$.
2. *If* $M\in\mathcal{M}$ *and* $\mathrm{fv}(M)=\{x_1^{L_1},\dots,x_n^{L_n}\}$, *we denote* env_M^ω *the type environment* $(x_i^{L_i}:\omega^{L_i})_n$.
3. *We say that a type environment* Γ *is OK (and write* $\mathrm{OK}(\Gamma)$*) iff for all* $x^L:U\in\Gamma$, $d(U)=L$.
4. *Let* $\Gamma_1=(x_i^{L_i}:U_i)_n,\Gamma_1'$ *and* $\Gamma_2=(x_i^{L_i}:U_i')_n,\Gamma_2'$ *such that* $\mathrm{dom}(\Gamma_1')\cap\mathrm{dom}(\Gamma_2')=\emptyset$ *and for all* $i\in\{1,\dots,n\}$, $d(U_i)=d(U_i')$. *We denote* $\Gamma_1\sqcap\Gamma_2$ *the type environment* $(x_i^{L_i}:U_i\sqcap U_i')_n,\Gamma_1',\Gamma_2'$. *Note that* $\Gamma_1\sqcap\Gamma_2$ *is a type environment,* $\mathrm{dom}(\Gamma_1\sqcap\Gamma_2)=\mathrm{dom}(\Gamma_1)\cup\mathrm{dom}(\Gamma_2)$ *and that, on environments,* $\sqcap$ *is commutative, associative and idempotent.*
5. *Let* $\Gamma=(x_i^{L_i}:U_i)_{1\le i\le n}$ *We denote* $\overline{e}_j\Gamma=(x_i^{j::L_i}:\overline{e}_jU_i)_{1\le i\le n}$.
Note that $e\Gamma$ *is a type environment and* $e(G_1\sqcap\Gamma_2)=e\Gamma_1\sqcap e\Gamma_2$.
6. *We write* $\Gamma_1\diamond\Gamma_2$ *iff* $x^L\in\mathrm{dom}(\Gamma_1)$ *and* $x^K\in\mathrm{dom}(\Gamma_2)$ *implies* $K=L$.
7. *We follow [3] and write type judgements as* $M:\langle\Gamma\vdash U\rangle$ *instead of the traditional format of* $\Gamma\vdash M:U$, *where* $\vdash$ *is our typing relation. The typing rules of* $\vdash$ *are given on the left hand side of Figure 7. In the last clause, the binary relation* $\sqsubseteq$ *is defined on* $\mathbb{U}$ *by the rules on the right hand side of Figure 7. We let* Φ *denote types in* $\mathbb{U}$, *or environments* Γ *or typings* $\langle\Gamma\vdash U\rangle$. *When* $\Phi\sqsubseteq\Phi'$, *then* Φ *and* Φ' *belong to the same set (*$\mathbb{U}$*/environments/typings).*
8. *If* $L\in\mathcal{L}_\mathbb{N}$, $U\in\mathbb{U}$ *and* $\Gamma=(x_i^{L_i}:U_i)_n$ *is a type environment, we say that:*
   - $d(\Gamma)\succeq L$ *if and only if for all* $i\in\{1,\dots,n\}$, $d(U_i)\succeq L$ *and* $L_i\succeq L$.
   - $d(\langle\Gamma\vdash U\rangle)\succeq L$ *if and only if* $d(\Gamma)\succeq L$ *and* $d(U)\succeq L$.

To illustrate how our indexed type system works, we give an example:

Example 1. Let $U=\overline{e}_3(\overline{e}_2(\overline{e}_1((\overline{e}_0b\to c)\to(\overline{e}_0(a\sqcap(a\to b))\to c))\to d)\to(((\overline{e}_2d\to a)\sqcap b)\to a))$ where $a,b,c,d\in\mathcal{A}$,
$L_1=3::\oslash\preceq L_2=3::2::\oslash\preceq L_3=3::2::1::0::\oslash$
and
$M=\lambda x^{L_2}.\lambda y^{L_1}.(y^{L_1}\,(x^{L_2}\,\lambda u^{L_3}.\lambda v^{L_3}.(u^{L_3}\,(v^{L_3}\,v^{L_3})))).$
We invite the reader to check that $M:\langle()\vdash U\rangle$.

Just as we did for terms, we decrease the indexes of types, environments and typings.

Definition 7. 1. *If* $d(U)\succeq L$, *then if* $L=\oslash$ *then* $U^{-L}=U$ *else* $L=i::K$ *and we inductively define the type* U^{-L} *as follows:*
$(U_1\sqcap U_2)^{-i::K}=U_1^{-i::K}\sqcap U_2^{-i::K}$ $\qquad(\overline{e}_iU)^{-i::K}=U^{-K}$
We write U^{-i} *instead of* $U^{-(i)}$.

Typing rules	Subtyping rules
$\dfrac{}{x^{\oslash} : \langle (x^{\oslash} : T) \vdash T \rangle}\ (ax)$	$\dfrac{}{\Phi \sqsubseteq \Phi}\ (ref)$
$\dfrac{}{M : \langle env^{\omega}_{M} \vdash \omega^{\mathrm{d}(M)} \rangle}\ (\omega)$	$\dfrac{\Phi_1 \sqsubseteq \Phi_2 \quad \Phi_2 \sqsubseteq \Phi_3}{\Phi_1 \sqsubseteq \Phi_3}\ (tr)$
$\dfrac{M : \langle \Gamma, (x^L : U) \vdash T \rangle}{\lambda x^L.M : \langle \Gamma \vdash U \to T \rangle}\ (\to_I)$	$\dfrac{d(U_1) = d(U_2)}{U_1 \sqcap U_2 \sqsubseteq U_1}\ (\sqcap_E)$
$\dfrac{M : \langle \Gamma \vdash T \rangle \quad x^L \notin \mathrm{dom}(\Gamma)}{\lambda x^L.M : \langle \Gamma \vdash \omega^L \to T \rangle}\ (\to'_I)$	$\dfrac{U_1 \sqsubseteq V_1 \quad U_2 \sqsubseteq V_2}{U_1 \sqcap U_2 \sqsubseteq V_1 \sqcap V_2}\ (\sqcap)$
$\dfrac{M_1 : \langle \Gamma_1 \vdash U \to T \rangle \quad M_2 : \langle \Gamma_2 \vdash U \rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1 M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T \rangle}\ (\to_E)$	$\dfrac{U_2 \sqsubseteq U_1 \quad T_1 \sqsubseteq T_2}{U_1 \to T_1 \sqsubseteq U_2 \to T_2}\ (\to)$
$\dfrac{M : \langle \Gamma \vdash U_1 \rangle \qquad M : \langle \Gamma \vdash U_2 \rangle}{M : \langle \Gamma \vdash U_1 \sqcap U_2 \rangle}\ (\sqcap_I)$	$\dfrac{U_1 \sqsubseteq U_2}{eU_1 \sqsubseteq eU_2}\ (\sqsubseteq_e)$
$\dfrac{M : \langle \Gamma \vdash U \rangle}{M^{+j} : \langle \overline{e}_j \Gamma \vdash \overline{e}_j U \rangle}\ (e)$	$\dfrac{U_1 \sqsubseteq U_2}{\Gamma, y^L : U_1 \sqsubseteq \Gamma, y^L : U_2}\ (\sqsubseteq_c)$
$\dfrac{M : \langle \Gamma \vdash U \rangle \qquad \langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle}{M : \langle \Gamma' \vdash U' \rangle}\ (\sqsubseteq)$	$\dfrac{U_1 \sqsubseteq U_2 \quad \Gamma_2 \sqsubseteq \Gamma_1}{\langle \Gamma_1 \vdash U_1 \rangle \sqsubseteq \langle \Gamma_2 \vdash U_2 \rangle}\ (\sqsubseteq_{\langle\rangle})$

Fig. 1. Typing rules / Subtyping rules

2. *If $\Gamma = (x_i^{L_i} : U_i)_k$ and $d(\Gamma) \succeq L$, then for all $i \in \{1, \ldots, k\}$, $L_i = L :: L'_i$ and $d(U_i) \succeq L$ and we denote $\Gamma^{-L} = (x^{L'_i} : U_i^{-L})_k$.*
   We write Γ^{-i} instead of $\Gamma^{-(i)}$.
3. *If U is a type and Γ is a type environment such that $d(\Gamma) \succeq K$ and $d(U) \succeq K$, then we denote $(\langle \Gamma \vdash U \rangle)^{-K} = \langle \Gamma^{-K} \vdash U^{-K} \rangle$.*

The next lemma is informative about types and their degrees.

Lemma 2. *1. If $T \in \mathbb{T}$, then $d(T) = \oslash$.*

2. *Let $U \in \mathbb{U}$. If $d(U) = L = (n_i)_m$, then $U = \omega^L$ or $U = \boldsymbol{e}_L \sqcap_{i=1}^{p} T_i$ where $p \geq 1$ and for all $i \in \{1, \ldots, p\}$, $T_i \in \mathbb{T}$.*
3. *Let $U_1 \sqsubseteq U_2$.*
   (a) *$d(U_1) = d(U_2)$.*
   (b) *If $U_1 = \omega^K$ then $U_2 = \omega^K$.*
   (c) *If $U_1 = \boldsymbol{e}_K U$ then $U_2 = \boldsymbol{e}_K U'$ and $U \sqsubseteq U'$.*
   (d) *If $U_2 = \boldsymbol{e}_K U$ then $U_1 = \boldsymbol{e}_K U'$ and $U \sqsubseteq U'$.*
   (e) *If $U_1 = \sqcap_{i=1}^{p} \boldsymbol{e}_K (U_i \to T_i)$ where $p \geq 1$ then $U_2 = \omega^K$ or $U_2 = \sqcap_{j=1}^{q} \boldsymbol{e}_K (U'_j \to T'_j)$ where $q \geq 1$ and for all $j \in \{1, \ldots, q\}$, there exists $i \in \{1, \ldots, p\}$ such that $U'_j \sqsubseteq U_i$ and $T_i \sqsubseteq T'_j$.*
4. *If $U \in \mathbb{U}$ such that $d(U) = L$ then $U \sqsubseteq \omega^L$.*
5. *If $U \sqsubseteq U'_1 \sqcap U'_2$ then $U = U_1 \sqcap U_2$ where $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$.*

6. *If* $\Gamma \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2$ *then* $\Gamma = \Gamma_1 \sqcap \Gamma_2$ *where* $\Gamma_1 \sqsubseteq \Gamma'_1$ *and* $\Gamma_2 \sqsubseteq \Gamma'_2$.

The next lemma says how ordering or the decreasing of indexes propagate to environments.

Lemma 3. *1.* $\mathrm{OK}(env^{\omega}_M)$.
2. *If* $\Gamma \sqsubseteq \Gamma'$, $U \sqsubseteq U'$ *and* $x^L \notin \mathrm{dom}(\Gamma)$ *then* $\Gamma, (x^L : U) \sqsubseteq \Gamma', (x^L : U')$.
3. $\Gamma \sqsubseteq \Gamma'$ *iff* $\Gamma = (x_i^{L_i} : U_i)_n$, $\Gamma' = (x_i^{L_i} : U'_i)_n$ *and for every* $1 \leq i \leq n$, $U_i \sqsubseteq U'_i$.
4. $\langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle$ *iff* $\Gamma' \sqsubseteq \Gamma$ *and* $U \sqsubseteq U'$.
5. *If* $\mathrm{dom}(\Gamma) = \mathrm{fv}(M)$ *and* $\mathrm{OK}(\Gamma)$ *then* $\Gamma \sqsubseteq env^{\omega}_M$
6. *If* $\Gamma \diamond \Delta$ *and* $d(\Gamma), d(\Delta) \succeq K$, *then* $\Gamma^{-K} \diamond \Delta^{-K}$.
7. *If* $U \sqsubseteq U'$ *and* $d(U) \succeq K$ *then* $U^{-K} \sqsubseteq U'^{-K}$.
8. *If* $\Gamma \sqsubseteq \Gamma'$ *and* $d(\Gamma) \succeq K$ *then* $\Gamma^{-K} \sqsubseteq \Gamma'^{-K}$.
9. *If* $\mathrm{OK}(\Gamma_1)$, $\mathrm{OK}(\Gamma_2)$ *then* $\mathrm{OK}(\Gamma_1 \sqcap \Gamma_2)$.
10. *If* $\mathrm{OK}(\Gamma)$ *then* $\mathrm{OK}(e\Gamma)$.
11. *If* $\Gamma_1 \sqsubseteq \Gamma_2$ *then* $(d(\Gamma_1) \succeq L$ *iff* $d(\Gamma_2) \succeq L)$ *and* $(\mathrm{OK}(\Gamma_1)$ *iff* $\mathrm{OK}(\Gamma_2))$.

The next lemma shows that we do not allow weakening in $\vdash$.

Lemma 4. *1. For every* Γ *and* M *such that* $\mathrm{OK}(\Gamma)$ $\mathrm{dom}(\Gamma) = \mathrm{fv}(M)$ *and* $d(M) = K$, *we have* $M : \langle \Gamma \vdash \omega^K \rangle$.
2. *If* $M : \langle \Gamma \vdash U \rangle$, *then* $\mathrm{dom}(\Gamma) = \mathrm{fv}(M)$.
3. *If* $M_1 : \langle \Gamma_1 \vdash U \rangle$ *and* $M_2 : \langle \Gamma_2 \vdash V \rangle$ *then* $\Gamma_1 \diamond \Gamma_2$ *iff* $M_1 \diamond M_2$.

Proof. 1. By ω, $M : \langle env^{\omega}_M \vdash \omega^K \rangle$. By Lemma 3.5, $\Gamma \sqsubseteq env^{\omega}_M$. Hence, by $\sqsubseteq$ and $\sqsubseteq_{\langle\rangle}$, $M : \langle \Gamma \vdash \omega^K \rangle$.
2. By induction on the derivation $M : \langle \Gamma \vdash U \rangle$.
3. If) Let $x^L \in \mathrm{dom}(\Gamma_1)$ and $x^K \in \mathrm{dom}(\Gamma_2)$ then by Lemma 4.2, $x^L \in \mathrm{fv}(M_1)$ and $x^K \in \mathrm{fv}(M_2)$ so $\Gamma_1 \diamond \Gamma_2$. Only if) Let $x^L \in \mathrm{fv}(M_1)$ and $x^K \in \mathrm{fv}(M_2)$ then by Lemma 4.2, $x^L \in \mathrm{dom}(\Gamma_1)$ and $x^K \in \mathrm{dom}(\Gamma_2)$ so $M_1 \diamond M_2$. □

The next theorem states that typings are well defined and that within a typing, degrees are well behaved.

Theorem 3. *1. The typing relation* $\vdash$ *is well defined on* $\mathcal{M} \times Env \times \mathbb{U}$.
2. *If* $M : \langle \Gamma \vdash U \rangle$ *then* $\mathrm{OK}(\Gamma)$, *and* $d(\Gamma) \succeq d(U) = d(M)$.
3. *If* $M : \langle \Gamma \vdash U \rangle$ *and* $d(U) \succeq K$ *then* $M^{-K} : \langle \Gamma^{-K} \vdash U^{-K} \rangle$.

Proof. We prove 1. and 2. simultaneously by induction on the derivation $M : \langle \Gamma \vdash U \rangle$. We prove 3. by induction on the derivation $M : \langle \Gamma \vdash U \rangle$. Full details can be found in [12]. □

Finally, here are two derivable typing rules that we will freely use in the rest of the article.

Remark 1. 1. The rule $\dfrac{M : \langle \Gamma_1 \vdash U_1 \rangle \qquad M : \langle \Gamma_2 \vdash U_2 \rangle}{M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U_1 \sqcap U_2 \rangle}\ \sqcap'_I$ is derivable.
2. The rule $\dfrac{}{x^{\mathrm{d}(U)} : \langle (x^{\mathrm{d}(U)} : U) \vdash U \rangle}\ ax'$ is derivable.

4 Subject reduction properties

In this section we show that subject reduction holds for $\vdash$. The proof of subject reduction uses generation and substitution. Hence the next two lemmas.

Lemma 5 (Generation for $\vdash$).

1. *If $x^L : \langle \Gamma \vdash U \rangle$, then $\Gamma = (x^L : V)$ and $V \sqsubseteq U$.*
2. *If $\lambda x^L.M : \langle \Gamma \vdash U \rangle$, $x^L \in \mathrm{fv}(M)$ and $d(U) = K$, then $U = \omega^K$ or $U = \sqcap_{i=1}^p \boldsymbol{e}_K(V_i \to T_i)$ where $p \geq 1$ and for all $i \in \{1, \ldots, p\}$, $M : \langle \Gamma, x^L : \boldsymbol{e}_K V_i \vdash \boldsymbol{e}_K T_i \rangle$.*
3. *If $\lambda x^L.M : \langle \Gamma \vdash U \rangle$, $x^L \notin \mathrm{fv}(M)$ and $d(U) = K$, then $U = \omega^K$ or $U = \sqcap_{i=1}^p \boldsymbol{e}_K(V_i \to T_i)$ where $p \geq 1$ and for all $i \in \{1, \ldots, p\}$, $M : \langle \Gamma \vdash \boldsymbol{e}_K T_i \rangle$.*
4. *If $M\ x^L : \langle \Gamma, (x^L : U) \vdash T \rangle$ and $x^L \notin \mathrm{fv}(M)$, then $M : \langle \Gamma \vdash U \to T \rangle$.*

Lemma 6 (Substitution for $\vdash$). *If $M : \langle \Gamma, x^L : U \vdash V \rangle$, $N : \langle \Delta \vdash U \rangle$ and $M \diamond N$ then $M[x^L := N] : \langle \Gamma \sqcap \Delta \vdash V \rangle$.*

Since $\vdash$ does not allow weakening, we need the next definition since when a term is reduced, it may lose some of its free variables and hence will need to be typed in a smaller environment.

Definition 8. *If Γ is a type environment and $\mathcal{U} \subseteq \mathrm{dom}(\Gamma)$, then we write $\Gamma \upharpoonright_{\mathcal{U}}$ for the restriction of Γ on the variables of $\mathcal{U}$. If $\mathcal{U} = \mathrm{fv}(M)$ for a term M, we write $\Gamma \upharpoonright_M$ instead of $\Gamma \upharpoonright_{\mathrm{fv}(M)}$.*

Now we are ready to prove the main result of this section:

Theorem 4 (Subject reduction for $\vdash$). *If $M : \langle \Gamma \vdash U \rangle$ and $M \rhd^*_{\beta\eta} N$, then $N : \langle \Gamma \upharpoonright_N \vdash U \rangle$.*

Proof. By induction on the length of the derivation $M \rhd^*_{\beta\eta} N$. Case $M \rhd_{\beta\eta} N$ is by induction on the derivation $M : \langle \Gamma \vdash_3 U \rangle$. □

Corollary 1. *1. If $M : \langle \Gamma \vdash U \rangle$ and $M \rhd^*_{\beta} N$, then $N : \langle \Gamma \upharpoonright_N \vdash U \rangle$.*
2. If $M : \langle \Gamma \vdash U \rangle$ and $M \rhd^*_{h} N$, then $N : \langle \Gamma \upharpoonright_N \vdash U \rangle$.

5 Subject expansion properties

In this section we show that subject β-expansion holds for $\vdash$ but that subject η-expansion fails.

The next lemma is needed for expansion.

Lemma 7. *If $M[x^L := N] : \langle \Gamma \vdash U \rangle$ and $x^L \in \mathrm{fv}(M)$ then there exist a type V and two type environments Γ_1, Γ_2 such that:*
$M : \langle \Gamma_1, x^L : V \vdash U \rangle \qquad N : \langle \Gamma_2 \vdash V \rangle \qquad \Gamma = \Gamma_1 \sqcap \Gamma_2$

Since more free variables might appear in the β-expansion of a term, the next definition gives a possible enlargement of an environment.

Definition 9. *Let $m \geq n$, $\Gamma = (x_i^{L_i} : U_i)_n$ and $\mathcal{U} = \{x_1^{L_1}, ..., x_m^{L_m}\}$. We write $\Gamma{\uparrow}^{\mathcal{U}}$ for $x_1^{L_1} : U_1, ..., x_n^{L_n} : U_n, x_{n+1}^{L_{n+1}} : \omega^{L_{n+1}}, ..., x_m^{L_m} : \omega^{L_m}$. Note that $\Gamma{\uparrow}^{\mathcal{U}}$ is a type environment. If $\mathrm{dom}(\Gamma) \subseteq \mathrm{fv}(M)$, we write $\Gamma{\uparrow}^M$ instead of $\Gamma{\uparrow}^{\mathrm{fv}(M)}$.*

We are now ready to establish that subject expansion holds for β (next theorem) and that it fails for η (Lemma 8).

Theorem 5 (Subject expansion for β). *If $N : \langle \Gamma \vdash U \rangle$ and $M \rhd_\beta^* N$, then $M : \langle \Gamma{\uparrow}^M \vdash U \rangle$.*

Proof. By induction on the length of the derivation $M \rhd_\beta^* N$ using the fact that if $\mathrm{fv}(P) \subseteq \mathrm{fv}(Q)$, then $(\Gamma{\uparrow}^P){\uparrow}^Q = \Gamma{\uparrow}^Q$. □

Corollary 2. *If $N : \langle \Gamma \vdash U \rangle$ and $M \rhd_h^* N$, then $M : \langle \Gamma{\uparrow}^M \vdash U \rangle$.*

Lemma 8 (Subject expansion fails for η). *Let a be an element of $\mathcal{A}$. We have:*

1. $\lambda y^{\oslash}.\lambda x^{\oslash}.y^{\oslash} x^{\oslash} \rhd_\eta \lambda y^{\oslash}.y^{\oslash}$
2. $\lambda y^{\oslash}.y^{\oslash} : \langle () \vdash a \to a \rangle$.
3. *It is not possible that*
   $\lambda y^{\oslash}.\lambda x^{\oslash}.y^{\oslash} x^{\oslash} : \langle () \vdash a \to a \rangle$.
   Hence, the subject η-expansion lemmas fail for $\vdash$.

Proof. 1. and 2. are easy. For 3., assume $\lambda y^{\oslash}.\lambda x^{\oslash}.y^{\oslash} x^{\oslash} : \langle () \vdash a \to a \rangle$.
By Lemma 5.2, $\lambda x^{\oslash}.y^{\oslash} x^{\oslash} : \langle (y : a) \vdash\to a \rangle$. Again, by Lemma 5.2, $a = \omega^{\oslash}$ or there exists $n \geq 1$ such that $a = \sqcap_{i=1}^n (U_i \to T_i)$, absurd. □

6 The realisability semantics

In this section we introduce the realisability semantics and show its soundness for $\vdash$.

Crucial to a realisability semantics is the notion of a saturated set:

Definition 10. *Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}$.*

1. *We use $\mathcal{P}(\mathcal{X})$ to denote the powerset of $\mathcal{X}$, i.e. $\{\mathcal{Y} \ / \ \mathcal{Y} \subseteq \mathcal{X}\}$.*
2. *We define $\mathcal{X}^{+i} = \{M^{+i} \ / \ M \in \mathcal{X}\}$.*
3. *We define $\mathcal{X} \rightsquigarrow \mathcal{Y} = \{M \in \mathcal{M} \ / \ M\ N \in \mathcal{Y}$ for all $N \in \mathcal{X}$ such that $M \diamond N\}$.*
4. *We say that $\mathcal{X} \wr \mathcal{Y}$ iff for all $M \in \mathcal{X} \rightsquigarrow \mathcal{Y}$, there exists $N \in \mathcal{X}$ such that $M \diamond N$.*
5. *For $r \in \{\beta, \beta\eta, h\}$, we say that $\mathcal{X}$ is r-saturated if whenever $M \rhd_r^* N$ and $N \in \mathcal{X}$, then $M \in \mathcal{X}$.*

Saturation is closed under intersection, lifting and arrows:

Lemma 9. 1. $(\mathcal{X} \cap \mathcal{Y})^{+i} = \mathcal{X}^{+i} \cap \mathcal{Y}^{+i}$.
2. *If $\mathcal{X}, \mathcal{Y}$ are r-saturated sets, then $\mathcal{X} \cap \mathcal{Y}$ is r-saturated.*
3. *If $\mathcal{X}$ is r-saturated, then $\mathcal{X}^{+i}$ is r-saturated.*

4. If $\mathcal{Y}$ is r-saturated, then, for every set $\mathcal{X}$, $\mathcal{X} \rightsquigarrow \mathcal{Y}$ is r-saturated.
5. $(\mathcal{X} \rightsquigarrow \mathcal{Y})^{+i} \subseteq \mathcal{X}^{+i} \rightsquigarrow \mathcal{Y}^{+i}$.
6. If $\mathcal{X}^{+i} \wr \mathcal{Y}^{+i}$, then $\mathcal{X}^{+i} \rightsquigarrow \mathcal{Y}^{+i} \subseteq (\mathcal{X} \rightsquigarrow \mathcal{Y})^{+i}$.

We now give the basic step in our realisability semantics: the interpretations and meanings of types.

Definition 11. *Let $\mathcal{V}_1$, $\mathcal{V}_2$ be countably infinite, $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ and $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$.*

1. *Let $L \in \mathcal{L}_{\mathbb{N}}$. We define $\mathcal{M}^L = \{M \in \mathcal{M} \ / \ d(M) = L\}$.*
2. *Let $x \in \mathcal{V}_1$. We define $\mathcal{N}_x^L = \{x^L \ N_1 ... N_k \in \mathcal{M} \ / \ k \geq 0\}$.*
3. *Let $r \in \{\beta, \beta\eta, h\}$. An r-interpretation $\mathcal{I} : \mathcal{A} \mapsto \mathcal{P}(\mathcal{M}^{\oslash})$ is a function such that for all $a \in \mathcal{A}$:*
   - *$\mathcal{I}(a)$ is r-saturated and • $\forall x \in \mathcal{V}_1. \ \mathcal{N}_x^{\oslash} \subseteq \mathcal{I}(a)$.*

   We extend an r-interpretation $\mathcal{I}$ to $\mathbb{U}$ as follows:
   - *$\mathcal{I}(\omega^L) = \mathcal{M}^L$ • $\mathcal{I}(\overline{e}_i U) = \mathcal{I}(U)^{+i}$*
   - *$\mathcal{I}(U_1 \sqcap U_2) = \mathcal{I}(U_1) \cap \mathcal{I}(U_2)$ • $\mathcal{I}(U \to T) = \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T)$*

   Let r-int $= \{\mathcal{I} \ / \ \mathcal{I}$ is an r-interpretation$\}$.
4. *Let $U \in \mathbb{U}$ and $r \in \{\beta, \beta\eta, h\}$. Define $[U]_r$, the r-interpretation of U by: $[U]_r = \{M \in \mathcal{M} \ / \ M$ is closed and $M \in \bigcap_{\mathcal{I} \in r\text{-}int} \mathcal{I}(U)\}$*

Lemma 10. *Let $r \in \{\beta, \beta\eta, h\}$.*

1. *(a) For any $U \in \mathbb{U}$ and $\mathcal{I} \in r$-int, we have $\mathcal{I}(U)$ is r-saturated.*
   (b) If $d(U) = L$ and $\mathcal{I} \in r$-int, then for all $x \in \mathcal{V}_1$, $\mathcal{N}_x^L \subseteq \mathcal{I}(U) \subseteq \mathcal{M}^L$.
2. *Let $r \in \{\beta, \beta\eta, h\}$. If $\mathcal{I} \in r$-int and $U \sqsubseteq V$, then $\mathcal{I}(U) \subseteq \mathcal{I}(V)$.*

Here is the soundness lemma.

Lemma 11 (Soundness). *Let $r \in \{\beta, \beta\eta, h\}$, $M : \langle (x_j^{L_j} : U_j)_n \vdash U \rangle$, $\mathcal{I} \in r$-int and for all $j \in \{1, \ldots, n\}$, $N_j \in \mathcal{I}(U_j)$. If $M[(x_j^{L_j} := N_j)_n] \in \mathcal{M}$ then $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(U)$.*

Proof. By induction on the derivation $M : \langle (x_j^{L_j} : U_j)_n \vdash U \rangle$. □

Corollary 3. *Let $r \in \{\beta, \beta\eta, h\}$. If $M : \langle () \vdash U \rangle$, then $M \in [U]_r$.* □

Proof. By Lemma 11, $M \in \mathcal{I}(U)$ for any r-interpretation $\mathcal{I}$. By Lemma 4.2, $\mathrm{fv}(M) = \mathrm{dom}(()) = \emptyset$ and hence M is closed. Therefore, $M \in [U]_r$. □

Lemma 12 (The meaning of types is closed under type operations). *Let $r \in \{\beta, \beta\eta, h\}$. On $\mathbb{U}$, the following hold:*

1. *$[\overline{e}_i U]_r = [U]_r^{+i}$*
2. *$[U \sqcap V]_r = [U]_r \cap [V]_r$*
3. *If $\mathcal{I} \in r$-int and $U, V \in \mathbb{U}$, then $\mathcal{I}(U) \wr \mathcal{I}(V)$.*

Proof. 1. and 2. are easy. 3. Let $\mathrm{d}(U) = K$, $M \in \mathcal{I}(U) \rightsquigarrow \mathcal{I}(V)$ and $x \in \mathcal{V}_1$ such that for all L, $x^L \notin \mathrm{fv}(M)$, then $M \diamond x^K$ and by lemma 10.1b, $x^K \in \mathcal{I}(U)$. □

The next definition and lemma put the realisability semantics in use.

Definition 12 (Examples). *Let $a, b \in \mathcal{A}$ where $a \neq b$. We define:*

- $Id_0 = a \to a$, $Id_1 = \overline{e}_1(a \to a)$ *and* $Id'_1 = \overline{e}_1 a \to \overline{e}_1 a$.
- $D = (a \sqcap (a \to b)) \to b$.
- $Nat_0 = (a \to a) \to (a \to a)$, $Nat_1 = \overline{e}_1((a \to a) \to (a \to a))$, *and* $Nat'_0 = (\overline{e}_1 a \to a) \to (\overline{e}_1 a \to a)$.

Moreover, if M, N are terms and $n \in \mathbb{N}$, we define $(M)^n\ N$ by induction on n: $(M)^0\ N = N$ and $(M)^{m+1}\ N = M\ ((M)^m\ N)$.

Lemma 13. *1.* $[Id_0]_\beta = \{M \in \mathcal{M}^\oslash \ /\ M$ *is closed and* $M \rhd^*_\beta \lambda y^\oslash.y^\oslash\}$.
2. $[Id_1]_\beta = [Id'_1]_\beta = \{M \in \mathcal{M}^{(1)} \ /\ M$ *is closed and* $M \rhd^*_\beta \lambda y^{(1)}.y^{(1)}\}$. *(Note that* $Id'_1 \notin \mathbb{U}$.*)*
3. $[D]_\beta = \{M \in \mathcal{M}^\oslash \ /\ M$ *is closed and* $M \rhd^*_\beta \lambda y^\oslash.y^\oslash y^\oslash\}$.
4. $[Nat_0]_\beta = \{M \in \mathcal{M}^\oslash \ /\ M$ *is closed and* $M \rhd^*_\beta \lambda f^\oslash.f^\oslash$ *or* $M \rhd^*_\beta \lambda f^\oslash.\lambda y^\oslash.(f^\oslash)^n y^\oslash$ *where* $n \geq 1\}$.
5. $[Nat_1]_\beta = \{M \in \mathcal{M}^{(1)} \ /\ M$ *is closed and* $M \rhd^*_\beta \lambda f^{(1)}.f^{(1)}$ *or* $M \rhd^*_\beta \lambda f^{(1)}.\lambda x^{(1)}.(f^{(1)})^n y^{(1)}$ *where* $n \geq 1\}$. *(Note that* $Nat'_1 \notin \mathbb{U}$.*)*
6. $[Nat'_0]_\beta = \{M \in \mathcal{M}^\oslash \ /\ M$ *is closed and* $M \rhd^*_\beta \lambda f^\oslash.f^\oslash$ *or* $M \rhd^*_\beta \lambda f^\oslash.\lambda y^{(1)}.f^\oslash y^{(1)}\}$.

7 The completeness theorem

In this section we set out the machinery and prove that completeness holds for $\vdash$.

We need the following partition of the set of variables $\{y^L / y \in \mathcal{V}_2\}$.

Definition 13. *1. Let $L \in \mathcal{L}_\mathbb{N}$. We define $\mathbb{U}^L = \{U \in \mathbb{U} / d(U) = L\}$ and $\mathcal{V}^L = \{x^L / x \in \mathcal{V}_2\}$.*
2. *Let $U \in \mathbb{U}$. We inductively define a set of variables $\mathbb{V}_U$ as follows:*
   - *If $d(U) = \oslash$ then:*
     - *$\mathbb{V}_U$ is an infinite set of variables of degree $\oslash$.*
     - *If $U \neq V$ and $d(U) = d(V) = \oslash$, then $\mathbb{V}_U \cap \mathbb{V}_V = \emptyset$.*
     - $\bigcup_{U \in \mathbb{U}^\oslash} \mathbb{V}_U = \mathcal{V}^\oslash$.
   - *If $d(U) = L$, then we put $\mathbb{V}_U = \{y^L \ /\ y^\oslash \in \mathbb{V}_{U^{-L}}\}$.*

Lemma 14. *1. If $d(U), d(V) \succeq L$ and $U^{-L} = V^{-L}$, then $U = V$.*
2. *If $d(U) = L$, then $\mathbb{V}_U$ is an infinite subset of $\mathcal{V}^L$.*
3. *If $U \neq V$ and $d(U) = d(V) = L$, then $\mathbb{V}_U \cap \mathbb{V}_V = \emptyset$.*
4. $\bigcup_{U \in \mathbb{U}^L} \mathbb{V}_U = \mathcal{V}^L$.
5. *If $y^L \in \mathbb{V}_U$, then $y^{i::L} \in \mathbb{V}_{\overline{e}_i U}$.*
6. *If $y^{i::L} \in \mathbb{V}_U$, then $y^L \in \mathbb{V}_{U^{-i}}$.*

Proof. 1. If $L = (n_i)_m$, we have $U = \overline{e}_{n_1} \dots \overline{e}_{n_m} U'$ and $V = \overline{e}_{n_1} \dots \overline{e}_{n_m} V'$. Then $U^{-L} = U'$, $V^{-L} = V'$ and $U' = V'$. Thus $U = V$. 2. 3. and 4. By induction on L and using 1. 5. Because $(\overline{e}_i U)^{-i} = U$. 6. By definition. □

Our partition of the set $\mathcal{V}_2$ as above will enable us to give in the next definition useful infinite sets which will contain type environments that will play a crucial role in one particular type interpretation.

Definition 14. *1. Let $L \in \mathcal{L}_{\mathbb{N}}$. We denote $\mathbb{G}^L = \{(y^L : U) \;/\; U \in \mathbb{U}^L$ and $y^L \in \mathbb{V}_U\}$ and $\mathbb{H}^L = \bigcup_{K \succeq L} \mathbb{G}^K$. Note that $\mathbb{G}^L$ and $\mathbb{H}^L$ are not type environments because they are infinite sets.*

2. *Let $L \in \mathcal{L}_{\mathbb{N}}$, $M \in \mathcal{M}$ and $U \in \mathbb{U}$, we write:*
   - *$M : \langle \mathbb{H}^L \vdash U \rangle$ if there is a type environment $\Gamma \subset \mathbb{H}^L$ where $M : \langle \Gamma \vdash U \rangle$*
   - *$M : \langle \mathbb{H}^L \vdash^* U \rangle$ if $M \rhd^*_{\beta\eta} N$ and $N : \langle \mathbb{H}^L \vdash U \rangle$*

Lemma 15. *1. If $\Gamma \subset \mathbb{H}^L$ then* $\mathrm{OK}(\Gamma)$.

2. *If $\Gamma \subset \mathbb{H}^L$ then $\overline{e}_i\Gamma \subset \mathbb{H}^{i::L}$.*
3. *If $\Gamma \subset \mathbb{H}^{i::L}$ then $\Gamma^{-i} \subset \mathbb{H}^L$.*
4. *If $\Gamma_1 \subset \mathbb{H}^L$, $\Gamma_2 \subset \mathbb{H}^K$ and $L \preceq K$ then $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^L$.*

Proof. 1. Let $x^K : U \in \Gamma$ then $U \in \mathbb{U}^K$ and so $\mathrm{d}(U) = K$. 2. and 3. are by lemma 14. 4. First note that by 1., $\Gamma_1 \sqcap \Gamma_2$ is well defined. $\mathbb{H}^K \subseteq \mathbb{H}^L$. Let $(x^R : U_1 \sqcap U_2) \in \Gamma_1 \sqcap \Gamma_2$ where $(x^R : U_1) \in \Gamma_1 \subset \mathbb{H}^L$ and $(x^R : U_2) \in \Gamma_2 \subset \mathbb{H}^K \subseteq \mathbb{H}^L$, then $\mathrm{d}(U_1) = \mathrm{d}(U_2) = R$ and $x^R \in \mathbb{V}_{U_1} \cap \mathbb{V}_{U_2}$. Hence, by lemma 14, $U_1 = U_2$ and $\Gamma_1 \sqcap \Gamma_2 = \Gamma_1 \cup \Gamma_2 \subset \mathbb{H}^L$. □

For every $L \in \mathcal{L}_{\mathbb{N}}$, we define the set of terms of degree L which contain some free variable x^K where $x \in \mathcal{V}_1$ and $K \succeq L$.

Definition 15. *For every $L \in \mathcal{L}_{\mathbb{N}}$, let $\mathcal{O}^L = \{M \in \mathcal{M}^L \;/\; x^K \in \mathrm{fv}(M)$, $x \in \mathcal{V}_1$ and $K \succeq L\}$. It is easy to see that, for every $L \in \mathcal{L}_{\mathbb{N}}$ and $x \in \mathcal{V}_1$, $\mathcal{N}_x^L \subseteq \mathcal{O}^L$.*

Lemma 16. *1. $(\mathcal{O}^L)^{+i} = \mathcal{O}^{i::L}$.*

2. *If $y \in \mathcal{V}_2$ and $(My^K) \in \mathcal{O}^L$, then $M \in \mathcal{O}^L$*
3. *If $M \in \mathcal{O}^L$, $M \diamond N$ and $L \preceq K = d(N)$, then $MN \in \mathcal{O}^L$.*
4. *If $d(M) = L$, $L \preceq K$, $M \diamond N$ and $N \in \mathcal{O}^K$, then $MN \in \mathcal{O}^L$.*

The crucial interpretation $\mathbb{I}$ for the proof of completeness is given as follows:

Definition 16. *1. Let $\mathbb{I}_{\beta\eta}$ be the $\beta\eta$-interpretation defined by: for all type variables a, $\mathbb{I}_{\beta\eta}(a) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}^{\oslash} \;/\; M : \langle \mathbb{H}^{\oslash} \vdash^* a \rangle\}$.*

2. *Let $\mathbb{I}_\beta$ be the β-interpretation defined by: for all type variables a, $\mathbb{I}_\beta(a) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}^{\oslash} \;/\; M : \langle \mathbb{H}^{\oslash} \vdash a \rangle\}$.*
3. *Let $\mathbb{I}_h$ be the h-interpretation defined by: for all type variables a, $\mathbb{I}_h(a) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}^{\oslash} \;/\; M : \langle \mathbb{H}^{\oslash} \vdash a \rangle\}$.*

The next crucial lemma shows that $\mathbb{I}$ is an interpretation and that the interpretation of a type of order L contains terms of order L which are typable in these special environments which are parts of the infinite sets of Definition 14.

Lemma 17. *Let $r \in \{\beta\eta, \beta, h\}$ and $r' \in \{\beta, h\}$*

1. *If $\mathbb{I}_r \in r$-int and $a \in \mathcal{A}$ then $\mathbb{I}_r(a)$ is r-saturated and for all $x \in \mathcal{V}_1, \mathcal{N}_x^{\oslash} \subseteq \mathbb{I}_r(a)$.*

2. If $U \in \mathbb{U}$ *and* $d(U) = L$*, then* $\mathbb{I}_{\beta\eta}(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \;/\; M : \langle \mathbb{H}^L \vdash^* U\rangle\}$.
3. If $U \in \mathbb{U}$ *and* $d(U) = L$*, then* $\mathbb{I}_{r'}(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \;/\; M : \langle \mathbb{H}^L \vdash U\rangle\}$.

Proof. 1. We do two cases:
Case $r = \beta\eta$. It is easy to see that $\forall x \in \mathcal{V}_1, \mathcal{N}_x^{\oslash} \subseteq \mathcal{O}^{\oslash} \subseteq \mathbb{I}_{\beta\eta}(a)$. Now we show that $\mathbb{I}_{\beta\eta}(a)$ is $\beta\eta$-saturated. Let $M \rhd^*_{\beta\eta} N$ and $N \in \mathbb{I}_{\beta\eta}(a)$.

- If $N \in \mathcal{O}^{\oslash}$ then $N \in \mathcal{M}^{\oslash}$ and $\exists L$ and $x \in \mathcal{V}_1$ such that $x^L \in \mathrm{fv}(N)$. By theorem 1.2, $\mathrm{fv}(N) \subseteq \mathrm{fv}(M)$ and $\mathrm{d}(M) = \mathrm{d}(N)$, hence, $M \in \mathcal{O}^{\oslash}$
- If $N \in \{M \in \mathcal{M}^{\oslash} \;/\; M : \langle \mathbb{H}^{\oslash} \vdash^* a\rangle\}$ then $N \rhd^*_{\beta\eta} N'$ and $\exists \Gamma \subset \mathbb{H}^{\oslash}$, such that $N' : \langle \Gamma \vdash a\rangle$. Hence $M \rhd^*_{\beta\eta} N'$ and since by theorem 1.2, $\mathrm{d}(M) = \mathrm{d}(N')$, $M \in \{M \in \mathcal{M}^{\oslash} \;/\; M : \langle \mathbb{H}^{\oslash} \vdash^* a\rangle\}$.

Case $r = \beta$. It is easy to see that $\forall x \in \mathcal{V}_1, \mathcal{N}_x^{\oslash} \subseteq \mathcal{O}^{\oslash} \subseteq \mathbb{I}_{\beta}(a)$. Now we show that $\mathbb{I}_{\beta}(a)$ is β-saturated. Let $M \rhd^*_{\beta} N$ and $N \in \mathbb{I}_{\beta}(a)$.

- If $N \in \mathcal{O}^{\oslash}$ then $N \in \mathcal{M}^{\oslash}$ and $\exists L$ and $x \in \mathcal{V}_1$ such that $x^L \in \mathrm{fv}(N)$. By theorem 1.2, $\mathrm{fv}(N) \subseteq \mathrm{fv}(M)$ and $\mathrm{d}(M) = \mathrm{d}(N)$, hence, $M \in \mathcal{O}^{\oslash}$
- If $N \in \{M \in \mathcal{M}^{\oslash} \;/\; M : \langle \mathbb{H}^{\oslash} \vdash a\rangle\}$ then $\exists \Gamma \subset \mathbb{H}^{\oslash}$, such that $N : \langle \Gamma \vdash a\rangle$. By theorem 5, $M : \langle \Gamma\!\uparrow^M \vdash a\rangle$. Since by theorem 1.2, $\mathrm{fv}(N) \subseteq \mathrm{fv}(M)$, let $\mathrm{fv}(N) = \{x_1^{L_1}, \ldots, x_n^{L_n}\}$ and $\mathrm{fv}(M) = \mathrm{fv}(N) \cup \{x_{n+1}^{L_{n+1}}, \ldots, x_{n+m}^{L_{n+m}}\}$. So $\Gamma\!\uparrow^M = \Gamma, (x_{n+1}^{L_{n+1}} : \omega^{L_{n+1}}, \ldots, x_{n+m}^{L_{n+m}} : \omega^{L_{n+m}})$. $\forall n+1 \leq i \leq n+m$, let U_i such that $x_i^{L_i} \in \mathbb{V}_{U_i}$. Then $\Gamma, (x_{n+1}^{L_{n+1}} : U_{n+1}, \ldots, x_{n+m}^{L_{n+m}} : U_{n+m}) \subset \mathbb{H}^{\oslash}$ and by $\sqsubseteq$, $M : \langle \Gamma, (x_{n+1}^{L_{n+1}} : U_{n+1}, \ldots, x_{n+m}^{L_{n+m}} : U_{n+m}) \vdash a\rangle$. Thus $M : \langle \mathbb{H}^{\oslash} \vdash a\rangle$ and since by theorem 1.2, $\mathrm{d}(M) = \mathrm{d}(N)$, $M \in \{M \in \mathcal{M}^{\oslash} \;/\; M : \langle \mathbb{H}^{\oslash} \vdash a\rangle\}$.

2. By induction on U.

- $U = a$: By definition of $\mathbb{I}_{\beta\eta}$.
- $U = \omega^L$: By definition, $\mathbb{I}_{\beta\eta}(\omega^L) = \mathcal{M}^L$. Hence, $\mathcal{O}^L \cup \{M \in \mathcal{M}^L \;/\; M : \langle \mathbb{H}^L \vdash^* \omega^L\rangle\} \subseteq \mathbb{I}_{\beta\eta}(\omega^L)$.
  Let $M \in \mathbb{I}_{\beta\eta}(\omega^L)$ where $\mathrm{fv}(M) = \{x_1^{L_1}, ..., x_n^{L_n}\}$ then $M \in \mathcal{M}^L$. $\forall\, 1 \leq i \leq n$, let U_i the type such that $x_i^{L_i} \in \mathbb{V}_{U_i}$. Then $\Gamma = (x_i^{L_i} : U_i)_n \subset \mathbb{H}^L$. By lemma 4.1 and lemma 15, $M : \langle \Gamma \vdash \omega^L\rangle$. Hence $M : \langle \mathbb{H}^L \vdash \omega^L\rangle$. Therefore, $\mathbb{I}(\omega^L) \subseteq \{M \in \mathcal{M}^L \;/\; M : \langle \mathbb{H}^L \vdash^* \omega^L\rangle\}$.
  We deduce $\mathbb{I}_{\beta\eta}(\omega^L) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \;/\; M : \langle \mathbb{H}^L \vdash^* \omega^L\rangle\}$.
- $U = \overline{e}_i V$: $L = i :: K$ and $\mathrm{d}(V) = K$. By IH and lemma 16, $\mathbb{I}_{\beta\eta}(\overline{e}_i V) = (\mathbb{I}_{\beta\eta}(V))^{+i} = (\mathcal{O}^K \cup \{M \in \mathcal{M}^K \;/\; M : \langle \mathbb{H}^K \vdash^* V\rangle\})^{+i} =$
  $\mathcal{O}^L \cup (\{M \in \mathcal{M}^K \;/\; M : \langle \mathbb{H}^K \vdash^* V\rangle\})^{+i}$.
  - If $M \in \mathcal{M}^K$ and $M : \langle \mathbb{H}^K \vdash^* V\rangle$, then $M \rhd^*_{\beta\eta} N$ and $N : \langle \Gamma \vdash V\rangle$ where $\Gamma \subset \mathbb{H}^K$. By e, lemmas 19 and 15, $N^{+i} : \langle \overline{e}_i \Gamma \vdash \overline{e}_i V\rangle$, $M^{+i} \rhd^*_{\beta\eta} N^{+i}$ and $\overline{e}_i \Gamma \subset \mathbb{H}^L$. Thus $M^{+i} \in \mathcal{M}^L$ and $M^{+i} : \langle \mathbb{H}^L \vdash^* U\rangle$.
  - If $M \in \mathcal{M}^L$ and $M : \langle \mathbb{H}^L \vdash^* U\rangle$, then $M \rhd^*_{\beta\eta} N$ and $N : \langle \Gamma \vdash U\rangle$ where $\Gamma \subset \mathbb{H}^L$. By lemmas 19, 3, and 15, $M^{-i} \rhd^*_{\beta\eta} N^{-i}$, $N^{-i} : \langle \Gamma^{-i} \vdash V\rangle$ and $\Gamma^{-i} \subset \mathbb{H}^K$. Thus by lemma 19, $M = (M^{-i})^{+i}$ and $M^{-i} \in \{M \in \mathcal{M}^K \;/\; M : \langle \mathbb{H}^K \vdash^* V\rangle\}$.

Hence $(\{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash^* V\rangle\})^{+i} = \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U\rangle\}$ and $\mathbb{I}_{\beta\eta}(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U\rangle\}$.

– $U = U_1 \sqcap U_2$: By IH, $\mathbb{I}_{\beta\eta}(U_1 \sqcap U_2) = \mathbb{I}_{\beta\eta}(U_1) \cap \mathbb{I}_{\beta\eta}(U_2) = (\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U_1\rangle\}) \cap (\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U_2\rangle\}) = \mathcal{O}^L \cup (\{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U_1\rangle\} \cap \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U_2\rangle\})$.
  - If $M \in \mathcal{M}^L$, $M : \langle \mathbb{H}^L \vdash^* U_1\rangle$ and $M : \langle \mathbb{H}^L \vdash^* U_2\rangle$, then $M \rhd^*_{\beta\eta} N_1$, $M \rhd^*_{\beta\eta} N_2$, $N_1 : \langle \Gamma_1 \vdash U_1\rangle$ and $N_2 : \langle \Gamma_2 \vdash U_2\rangle$ where $\Gamma_1, \Gamma_2 \subset \mathbb{H}^L$. By confluence theorem 2 and subject reduction theorem 4, $\exists M'$ such that $M \rhd^*_{\beta\eta} M'$, $M' : \langle \Gamma_1 \upharpoonright_{M'} \vdash U_1\rangle$ and $M' : \langle \Gamma_2 \upharpoonright_{M'} \vdash U_2\rangle$. Hence by Remark 1 and lemma 1 and lemma 4.2 and lemma 25.2, $M' : \langle (\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{M'} \vdash U_1 \sqcap U_2\rangle$ and, by lemma 15, $(\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{M'} \subseteq \Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^L$. Thus $M : \langle \mathbb{H}^L \vdash^* U_1 \sqcap U_2\rangle$.
  - If $M \in \mathcal{M}^L$ and $M : \langle \mathbb{H}^L \vdash^* U_1 \sqcap U_2\rangle$, then $M \rhd^*_{\beta\eta} N$, $N : \langle \Gamma \vdash U_1 \sqcap U_2\rangle$ and $\Gamma \subset \mathbb{H}^L$. By $\sqsubseteq$, $N : \langle \Gamma \vdash U_1\rangle$ and $N : \langle \Gamma \vdash U_2\rangle$.
    Hence, $M : \langle \mathbb{H}^L \vdash^* U_1\rangle$ and $M : \langle \mathbb{H}^L \vdash^* U_2\rangle$.

  We deduce that $\mathbb{I}_{\beta\eta}(U_1 \sqcap T_2) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U_1 \sqcap U_2\rangle\}$.

– $U = V \to T$: Let $\mathrm{d}(T) = \oslash \preceq K = \mathrm{d}(V)$. By IH, $\mathbb{I}_{\beta\eta}(V) = \mathcal{O}^K \cup \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash^* V\rangle\}$ and $\mathbb{I}_{\beta\eta}(T) = \mathcal{O}^\oslash \cup \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash^* T\rangle\}$. Note that $\mathbb{I}_{\beta\eta}(V \to T) = \mathbb{I}_{\beta\eta}(V) \rightsquigarrow \mathbb{I}_{\beta\eta}(T)$.
  - Let $M \in \mathbb{I}_{\beta\eta}(V) \rightsquigarrow \mathbb{I}_{\beta\eta}(T)$ and, by lemma 14, let $y^K \in \mathbb{V}_V$ such that $\forall K, y^K \notin \mathrm{fv}(M)$. Then $M \diamond y^K$. By remark 1, $y^K : \langle (y^K : V) \vdash^* V\rangle$. Hence $y^K : \langle \mathbb{H}^K \vdash^* V\rangle$. Thus, $y^K \in \mathbb{I}_{\beta\eta}(V)$ and $My^K \in \mathbb{I}_{\beta\eta}(T)$.
    - If $My^K \in \mathcal{O}^\oslash$, then since $y \in \mathcal{V}_2$, by lemma 16, $M \in \mathcal{O}^\oslash$.
    - If $My^K \in \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash^* T\rangle\}$ then $My^K \rhd^*_{\beta\eta} N$ and $N : \langle \Gamma \vdash T\rangle$ such that $\Gamma \subset \mathbb{H}^\oslash$, hence, $\lambda y^K.My^K \rhd^*_{\beta\eta} \lambda y^K.N$. We have two cases:
      - If $y^K \in \mathrm{dom}(\Gamma)$, then $\Gamma = \Delta, (y^K : V)$ and by $\to_I$, $\lambda y^K.N : \langle \Delta \vdash V \to T\rangle$.
      - If $y^K \notin \mathrm{dom}(\Gamma)$, let $\Delta = \Gamma$. By $\to'_I$, $\lambda y^K.N : \langle \Delta \vdash \omega^K \to T\rangle$. By $\sqsubseteq$, since $\langle \Delta \vdash \omega^K \to T\rangle \sqsubseteq \langle \Delta \vdash V \to T\rangle$, we have $\lambda y^K.N : \langle \Delta \vdash V \to T\rangle$.

      Note that $\Delta \subset \mathbb{H}^\oslash$. Since $\lambda y^K.My^K \rhd^*_{\beta\eta} M$ and $\lambda y^K.My^K \rhd^*_{\beta\eta} \lambda y^K.N$, by theorem 2 and theorem 4, there is M' such that $M \rhd^*_{\beta\eta} M'$, $\lambda y^K.N \rhd^*_{\beta\eta} M'$, $M' : \langle \Delta \upharpoonright_{M'} \vdash V \to T\rangle$. Since $\Delta \upharpoonright_{M'} \subseteq \Delta \subset \mathbb{H}^\oslash$, $M : \langle \mathbb{H}^\oslash \vdash^* V \to T\rangle$.
  - Let $M \in \mathcal{O}^\oslash \cup \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash^* V \to T\rangle\}$ and $N \in \mathbb{I}_{\beta\eta}(V) = \mathcal{O}^K \cup \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash^* V\rangle\}$ such that $M \diamond N$. Then, $\mathrm{d}(N) = K \succeq \oslash = \mathrm{d}(M)$.
    - If $M \in \mathcal{O}^\oslash$, then, by lemma 16, $MN \in \mathcal{O}^\oslash$.
    - If $M \in \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash^* V \to T\rangle\}$, then
      - If $N \in \mathcal{O}^K$, then, by lemma 16, $MN \in \mathcal{O}^\oslash$.
      - If $N \in \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash^* V\rangle\}$ then $M \rhd^*_{\beta\eta} M_1$, $N \rhd^*_{\beta\eta} N_1$, $M_1 : \langle \Gamma_1 \vdash V \to T\rangle$ and $N_1 : \langle \Gamma_2 \vdash V\rangle$ where $\Gamma_1 \subset \mathbb{H}^\oslash$ and $\Gamma_2 \subset \mathbb{H}^K$. By lemma 19 and theorem 1, $MN \rhd^*_{\beta\eta} M_1N_1$ and, by $\to_E$ and lemma 4.3, $M_1N_1 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle$. By lemma 15, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^\oslash$. Therefore $MN : \langle \mathbb{H}^\oslash \vdash^* T\rangle$.

We deduce that $\mathbb{I}_{\beta\eta}(V \to T) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}^{\oslash} \ / \ M : \langle \mathbb{H}^{\oslash} \vdash^* V \to T\rangle\}$.

3. We only do the case $r = \beta$. By induction on U.

- $U = a$: By definition of $\mathbb{I}_\beta$.
- $U = \omega^L$: By definition, $\mathbb{I}_\beta(\omega^L) = \mathcal{M}^L$. Hence, $\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash \omega^L\rangle\} \subseteq \mathbb{I}_\beta(\omega^L)$.
  Let $M \in \mathbb{I}_\beta(\omega^L)$ where $\mathrm{fv}(M) = \{x_1^{L_1}, ..., x_n^{L_n}\}$ then $M \in \mathcal{M}^L$. $\forall\ 1 \le i \le n$, let U_i the type such that $x_i^{L_i} \in \mathbb{V}_{U_i}$. Then $\Gamma = (x_i^{L_i} : U_i)_n \subset \mathbb{H}^L$. By lemma 4.1 and lemma 15, $M : \langle \Gamma \vdash \omega^L\rangle$. Hence $M : \langle \mathbb{H}^L \vdash \omega^L\rangle$. Therefore, $\mathbb{I}(\omega^L) \subseteq \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash \omega^L\rangle\}$.
  We deduce $\mathbb{I}_\beta(\omega^L) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash \omega^L\rangle\}$.
- $U = \overline{e}_i V$: $L = i :: K$ and $\mathrm{d}(V) = K$. By IH and lemma 16, $\mathbb{I}_\beta(\overline{e}_i V) = (\mathbb{I}_\beta(V))^{+i} = (\mathcal{O}^K \cup \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash V\rangle\})^{+i} =$
  $\mathcal{O}^L \cup (\{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash V\rangle\})^{+i}$.
  - If $M \in \mathcal{M}^K$ and $M : \langle \mathbb{H}^K \vdash V\rangle$, then $M : \langle \Gamma \vdash V\rangle$ where $\Gamma \subset \mathbb{H}^K$. By e and 15, $M^{+i} : \langle \overline{e}_i\Gamma \vdash \overline{e}_i V\rangle$ and $\overline{e}_i\Gamma \subset \mathbb{H}^L$. Thus $M^{+i} \in \mathcal{M}^L$ and $M^{+i} : \langle \mathbb{H}^L \vdash U\rangle$.
  - If $M \in \mathcal{M}^L$ and $M : \langle \mathbb{H}^L \vdash U\rangle$, then $M : \langle \Gamma \vdash U\rangle$ where $\Gamma \subset \mathbb{H}^L$. By lemmas 3, and 15, $M^{-i} : \langle \Gamma^{-i} \vdash V\rangle$ and $\Gamma^{-i} \subset \mathbb{H}^K$. Thus by lemma 19, $M = (M^{-i})^{+i}$ and $M^{-i} \in \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash V\rangle\}$.

  Hence $(\{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash V\rangle\})^{+i} = \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U\rangle\}$ and $\mathbb{I}_\beta(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U\rangle\}$.
- $U = U_1 \sqcap U_2$: By IH, $\mathbb{I}_\beta(U_1 \sqcap U_2) = \mathbb{I}_\beta(U_1) \cap \mathbb{I}_\beta(U_2) = (\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U_1\rangle\}) \cap (\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U_2\rangle\}) = \mathcal{O}^L \cup (\{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U_1\rangle\} \cap \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U_2\rangle\})$.
  - If $M \in \mathcal{M}^L$, $M : \langle \mathbb{H}^L \vdash U_1\rangle$ and $M : \langle \mathbb{H}^L \vdash U_2\rangle$, then $M : \langle \Gamma_1 \vdash U_1\rangle$ and $M : \langle \Gamma_2 \vdash U_2\rangle$ where $\Gamma_1, \Gamma_2 \subset \mathbb{H}^L$. Hence by Remark 1, $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U_1 \sqcap U_2\rangle$ and, by lemma 15, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^L$. Thus $M : \langle \mathbb{H}^L \vdash U_1 \sqcap U_2\rangle$.
  - If $M \in \mathcal{M}^L$ and $M : \langle \mathbb{H}^L \vdash U_1 \sqcap U_2\rangle$, then $M : \langle \Gamma \vdash U_1 \sqcap U_2\rangle$ and $\Gamma \subset \mathbb{H}^L$. By $\sqsubseteq$, $M : \langle \Gamma \vdash U_1\rangle$ and $M : \langle \Gamma \vdash U_2\rangle$. Hence, $M : \langle \mathbb{H}^L \vdash U_1\rangle$ and $M : \langle \mathbb{H}^L \vdash U_2\rangle$.

  We deduce that $\mathbb{I}_\beta(U_1 \sqcap T_2) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U_1 \sqcap U_2\rangle\}$.
- $U = V \to T$: Let $\mathrm{d}(T) = \oslash \preceq K = \mathrm{d}(V)$. By IH, $\mathbb{I}_\beta(V) = \mathcal{O}^K \cup \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash V\rangle\}$ and $\mathbb{I}_\beta(T) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}^{\oslash} \ / \ M : \langle \mathbb{H}^{\oslash} \vdash T\rangle\}$. Note that $\mathbb{I}_\beta(V \to T) = \mathbb{I}_\beta(V) \rightsquigarrow \mathbb{I}_\beta(T)$.
  - Let $M \in \mathbb{I}_\beta(V) \rightsquigarrow \mathbb{I}_\beta(T)$ and, by lemma 14, let $y^K \in \mathbb{V}_V$ such that $\forall K, y^K \notin \mathrm{fv}(M)$. Then $M \diamond y^K$. By remark 1, $y^K : \langle (y^K : V) \vdash^* V\rangle$. Hence $y^K : \langle \mathbb{H}^K \vdash V\rangle$. Thus, $y^K \in \mathbb{I}_\beta(V)$ and $My^K \in \mathbb{I}_\beta(T)$.
    - If $My^K \in \mathcal{O}^{\oslash}$, then since $y \in \mathcal{V}_2$, by lemma 16, $M \in \mathcal{O}^{\oslash}$.
    - If $My^K \in \{M \in \mathcal{M}^{\oslash} \ / \ M : \langle \mathbb{H}^{\oslash} \vdash T\rangle\}$ then $My^K : \langle \Gamma \vdash T\rangle$ such that $\Gamma \subset \mathbb{H}^{\oslash}$. Since by lemma 4.2, $\mathrm{dom}(\Gamma) = \mathrm{fv}(My^K)$ and $y^K \in \mathrm{fv}(My^K)$, $\Gamma = \Delta, (y^K : V')$. Since $(y^K : V') \in \mathbb{H}^{\oslash}$, by lemma 14, $V = V'$. So $My^K : \langle \Delta, (y^K : V) \vdash T\rangle$ and by lemma 5 $M : \langle \Delta \vdash V \to T\rangle$. Note that $\Delta \subset \mathbb{H}^{\oslash}$, hence $M : \langle \mathbb{H}^{\oslash} \vdash V \to T\rangle$.

- Let $M \in \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}^{\oslash} \;/\; M : \langle \mathbb{H}^{\oslash} \vdash V \to T\rangle\}$ and $N \in \mathbb{I}_{\beta\eta}(V) = \mathcal{O}^{K} \cup \{M \in \mathcal{M}^{K} \;/\; M : \langle \mathbb{H}^{K} \vdash V\rangle\}$ such that $M \diamond N$. Then, $\mathrm{d}(N) = K \succeq \oslash = \mathrm{d}(M)$.
  - ∗ If $M \in \mathcal{O}^{\oslash}$, then, by lemma 16, $MN \in \mathcal{O}^{\oslash}$.
  - ∗ If $M \in \{M \in \mathcal{M}^{\oslash} \;/\; M : \langle \mathbb{H}^{\oslash} \vdash V \to T\rangle\}$, then
    - · If $N \in \mathcal{O}^{K}$, then, by lemma 16, $MN \in \mathcal{O}^{\oslash}$.
    - · If $N \in \{M \in \mathcal{M}^{K} \;/\; M : \langle \mathbb{H}^{K} \vdash V\rangle\}$ then $M : \langle \Gamma_1 \vdash V \to T\rangle$ and $N : \langle \Gamma_2 \vdash V\rangle$ where $\Gamma_1 \subset \mathbb{H}^{\oslash}$ and $\Gamma_2 \subset \mathbb{H}^{K}$. By $\to_E$ and lemma 4.3, $MN : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle$. By lemma 15, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^{\oslash}$. Therefore $MN : \langle \mathbb{H}^{\oslash} \vdash T\rangle$.

We deduce that $\mathbb{I}_{\beta}(V \to T) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}^{\oslash} \;/\; M : \langle \mathbb{H}^{\oslash} \vdash V \to T\rangle\}$. □

Now, we use this crucial $\mathbb{I}$ to establish completeness of our semantics.

Theorem 6 (Completeness of $\vdash$). *Let $U \in \mathbb{U}$ such that $d(U) = L$.*

1. $[U]_{\beta\eta} = \{M \in \mathcal{M}^{L} \;/\; M$ *closed*, $M \rhd^{*}_{\beta\eta} N$ *and* $N : \langle () \vdash U\rangle\}$.
2. $[U]_{\beta} = [U]_{h} = \{M \in \mathcal{M}^{L} \;/\; M : \langle () \vdash U\rangle\}$.
3. $[U]_{\beta\eta}$ *is stable by reduction. I.e., If* $M \in [U]_{\beta\eta}$ *and* $M \rhd^{*}_{\beta\eta} N$ *then* $N \in [U]_{\beta\eta}$.

Proof. Let $r \in \{\beta, h, \beta\eta\}$.

1. Let $M \in [U]_{\beta\eta}$. Then M is a closed term and $M \in \mathbb{I}_{\beta\eta}(U)$. Hence, by Lemma 17, $M \in \mathcal{O}^{L} \cup \{M \in \mathcal{M}^{L} \;/\; M : \langle \mathbb{H}^{L} \vdash^{*} U\rangle\}$. Since M is closed, $M \notin \mathcal{O}^{L}$. Hence, $M \in \{M \in \mathcal{M}^{L} \;/\; M : \langle \mathbb{H}^{L} \vdash^{*} U\rangle\}$ and so, $M \rhd^{*}_{\beta\eta} N$ and $N : \langle \Gamma \vdash U\rangle$ where $\Gamma \subset \mathbb{H}^{L}$. By Theorem 1, N is closed and, by Lemma 4.2, $N : \langle () \vdash U\rangle$. Conversely, take M closed such that $M \rhd^{*}_{\beta} N$ and $N : \langle () \vdash U\rangle$. Let $\mathcal{I} \in \beta\eta$-int. By Lemma 11, $N \in \mathcal{I}(U)$. By Lemma 10.1, $\mathcal{I}(U)$ is $\beta\eta$-saturated. Hence, $M \in \mathcal{I}(U)$. Thus $M \in [U]$.
2. Let $M \in [U]_{\beta}$. Then M is a closed term and $M \in \mathbb{I}_{\beta}(U)$. Hence, by Lemma 17, $M \in \mathcal{O}^{L} \cup \{M \in \mathcal{M}^{L} \;/\; M : \langle \mathbb{H}^{L} \vdash U\rangle\}$. Since M is closed, $M \notin \mathcal{O}^{L}$. Hence, $M \in \{M \in \mathcal{M}^{L} \;/\; M : \langle \mathbb{H}^{L} \vdash U\rangle\}$ and so, $M : \langle \Gamma \vdash U\rangle$ where $\Gamma \subset \mathbb{H}^{L}$. By Lemma 4.2, $M : \langle () \vdash U\rangle$.
   Conversely, take M such that $M : \langle () \vdash U\rangle$. By Lemma 4.2, M is closed. Let $\mathcal{I} \in \beta$-int. By Lemma 11, $M \in \mathcal{I}(U)$. Thus $M \in [U]_{\beta}$.
   It is easy to see that $[U]_{\beta} = [U]_{h}$.
3. Let $M \in [U]_{\beta\eta}$ and $M \rhd^{*}_{\beta\eta} N$. By 1, M is closed, $M \rhd^{*}_{\beta\eta} P$ and $P : \langle () \vdash U\rangle$. By confluence Theorem 2, there is Q such that $P \rhd^{*}_{\beta\eta} Q$ and $N \rhd^{*}_{\beta\eta} Q$. By subject reduction Theorem 4, $Q : \langle () \vdash U\rangle$. By Theorem 1, N is closed and, by 1, $N \in [U]_{\beta\eta}$. □

8 Conclusion

Expansion may be viewed to work like a multi-layered simultaneous substitution. Moreover, expansion is a crucial part of a procedure for calculating principal typings and helps support compositional type inference. Because the early definitions of expansion were complicated, expansion variables (E-variables) were introduced to simplify and mechanise expansion. The aim of this paper is to give a complete semantics for intersection type systems with expansion variables.

The only earlier attempt (see Kamareddine, Nour, Rahli and Wells [13]) at giving a semantics for expansion variables could only handle the λI-calculus, did not allow a universal type, and was incomplete in the presence of more than one expansion variable. This paper overcomes these difficulties and gives a complete semantics for an intersection type system with an arbitrary (possibly infinite) number of expansion variables using a calculus indexed with finite sequences of natural numbers.

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A Proofs of Section 2

The next lemma is needed in the proofs.

Lemma 18. *Let $M, M', N, N_1, \dots, N_n \in \mathcal{M}$.*

1. *$M \diamond M$ and if $M \diamond N$ then $N \diamond M$.*
2. *If $\mathrm{fv}(M) \subseteq \mathrm{fv}(M')$ and $M' \diamond N$ then $M \diamond N$.*
3. *If $M \diamond N$ and M' is a subterm of M then $M' \diamond N$.*
4. *If $d(M) = L$ and x^K occurs in M, then $K \succeq L$.*
5. *If $\mathcal{X} = \{M\} \cup \{N_i / 1 \leq i \leq n\}$, for all $i \in \{1, \dots, n\}$, $d(N_i) = L_i$ and $\diamond\mathcal{X}$ then $M[(x_i^{L_i} := N_i)_n] \in \mathcal{M}$ and $d(M[(x_i^{L_i} := N_i)_n]) = d(M)$.*
6. *If $\mathcal{X} = \{M, N\} \cup \{N_i / 1 \leq i \leq n\}$, for all $i \in \{1, \dots, n\}$, $d(N_i) = L_i$ and $\diamond\mathcal{X}$ then $M[(x_i^{L_i} := N_i)_n] \diamond N[(x_i^{L_i} := N_i)_n]$*

Proof. 1. First, we prove $M \diamond M$ by induction on M.
  - Let $M = x^L$ then it is trivial.
  - Let $M = \lambda x^L.N$ such that $N \in \mathcal{M}$ and $L \succeq \mathrm{d}(N)$. Let $y^K, y^{K'} \in \mathrm{fv}(M)$ then $y^K, y^{K'} \in \mathrm{fv}(N)$ and we conclude using IH on N.
  - Let $M = M_1 M_2$ such that $M_1, M_2 \in \mathcal{M}$, $\mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$ and $M_1 \diamond M_2$. Let $x^L, x^K \in \mathrm{fv}(M)$ then either $x^L, x^K \in \mathrm{fv}(M_1)$ and we conclude using IH on M_1. Or $x^L, x^K \in \mathrm{fv}(M_2)$ and we conclude using IH on M_2. Or $x^L \in \mathrm{fv}(M_1)$ and $x^K \in \mathrm{fv}(M_2)$ and we conclude using $M_1 \diamond M_2$.

  Let $M \diamond N$, we prove $N \diamond M$. It is trivial by definition.
2. Let $x^L \in \mathrm{fv}(M) \subseteq \mathrm{fv}(M')$ and $x^K \in \mathrm{fv}(N)$ then by hypothesis $K = L$.
3. By induction on M.
  - Case $M = x^L$ is trivial.
  - Case $M = \lambda x^L.P$ where $\forall K \in \mathcal{L}_{\mathbb{N}}, x^K \notin \mathrm{fv}(N)$. If $M' = M$ then nothing to prove. Else M' is a subterm of P. If we prove that $P \diamond N$ then we can use IH to get $M' \diamond N$. Hence, now we prove $P \diamond N$. Let $y \in \mathcal{V}$ such that $y^K \in \mathrm{fv}(P)$ and $y^{K'} \in \mathrm{fv}(N)$. Since $x^{K'} \notin \mathrm{fv}(N)$, then $x \neq y$ and $y^K \neq x^L$. Hence $y^K \in \mathrm{fv}(M)$ and since $M \diamond N$ then $K = K'$. Hence, $P \diamond N$.
  - Case $M = M_1 M_2$. Let $i \in \{1, 2\}$. First we prove that $M_i \diamond N$: let $x \in \mathcal{V}$, such that $x^L \in \mathrm{fv}(M_i)$ and $x^K \in \mathrm{fv}(N)$, then $x^L \in \mathrm{fv}(M)$ and so $L = K$. Now, if $M' = M$ then nothing to prove. Else
    - Either M' is a subterm of M_1 and so by IH, since $M_1 \diamond N$, $M' \diamond N$.
    - Or M' is a subterm of M_2 and so by IH, since $M_2 \diamond N$, $M' \diamond N$.
4. By induction on M.
  - If $M = x^K$ then $\mathrm{d}(M) = K$ and since $\succeq$ is an order relation, $K \succeq K$.
  - If $M = M_1 M_2$ then $\mathrm{d}(M) = \mathrm{d}(M_1)$. Let $L' = \mathrm{d}(M_2)$ so $L' \succeq L$. By IH, if x^K occurs in M_1 then $K \succeq L$ and if x^K occurs in M_2 then $K \succeq L'$. Since x^K occurs in M, $K \succeq L$.
  - If $M = \lambda x^{L_1}.M_1$ then $L_1 \succeq \mathrm{d}(M_1) = \mathrm{d}(\lambda x^{L_1}.M_1) = L$. If x^K occurs in M, then $x^K = x^{L_1}$ or x^K occurs in M_1. By IH, if x^K occurs in M_1 then $K \succeq L$.
5. By induction on M.

- If $M = y^K$ then if $y^K = x_i^{L_i}$, for $1 \leq i \leq n$, then $M[(x_i^{L_i} := N_i)_n] = N_i \in \mathcal{M}$ and $\mathrm{d}(M[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(N_i) = L_i = K$. Else, $M[(x_i^{L_i} := N_i)_n] = y^K \in \mathcal{M}$ and $\mathrm{d}(M[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(y^K)$.
- If $M = M_1M_2$ then $\mathrm{d}(M) = \mathrm{d}(M_1)$ and $M[(x_i^{L_i} := N_i)_n] = M_1[(x_i^{L_i} := N_i)_n]M_2[(x_i^{L_i} := N_i)_n]$. Since $\forall N \in \mathcal{X}, M \diamond N$, by 3., $\forall N \in \mathcal{X}, M_1 \diamond N$ and $M_2 \diamond N$. Since $M_1, M_2 \in \mathcal{M}$, by IH, $M_1[(x_i^{L_i} := N_i)_n], M_2[(x_i^{L_i} := N_i)_n] \in \mathcal{M}$, $\mathrm{d}(M_1[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(M_1)$ and $\mathrm{d}(M_2[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(M_2)$. Let $x^K \in \mathrm{fv}(M_1[(x_i^{L_i} := N_i)_n])$ and $x^{K'} \in \mathrm{fv}(M_2[(x_i^{L_i} := N_i)_n])$. If $x^K \in \mathrm{fv}(M_1)$ then by 3., $\diamond(\{M_1, M_2\} \cup \{N_i / 1 \leq i \leq n\})$ hence $K = K'$. Let $1 \leq i \leq n$. If $x^K \in \mathrm{fv}(N_i)$ then by 3., $\diamond(\{M_2\} \cup \{N_i / 1 \leq i \leq n\})$ hence $K = K'$. So $M_1[(x_i^{L_i} := N_i)_n] \diamond M_2[(x_i^{L_i} := N_i)_n]$. Furthermore, $\mathrm{d}(M_2[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(M_2) \succeq \mathrm{d}(M_1) = \mathrm{d}(M_1[(x_i^{L_i} := N_i)_n])$ hence $M_1[(x_i^{L_i} := N_i)_n]M_2[(x_i^{L_i} := N_i)_n] \in \mathcal{M}$ and $\mathrm{d}(M_1[(x_i^{L_i} := N_i)_n]M_2[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(M_1[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(M_1) = \mathrm{d}(M)$.
- If $M = \lambda y^K.M_1$ where $K \succeq \mathrm{d}(M_1)$ and $\forall 1 \leq i \leq n$, $y \neq x_i$ and $\forall K' \in \mathcal{L}_{\mathbb{N}}$, $y^{K'} \notin \mathrm{fv}(N_i) \cup \{x_i^{L_i}\}$ then $M[(x_i^{L_i} := N_i)_n] = \lambda y^K.M_1[(x_i^{L_i} := N_i)_n]$. Since $M_1 \in \mathcal{M}$, then by 3. and IH $M_1[(x_i^{L_i} := N_i)_n] \in \mathcal{M}$ and $\mathrm{d}(M_1[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(M_1)$. So $\lambda y^K.M_1[(x_i^{L_i} := N_i)_n] \in \mathcal{M}$ and $\mathrm{d}(\lambda y^K.M_1[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(M_1[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(M_1) = \mathrm{d}(M)$.

6. By 5., $M[(x_i^{L_i} := N_i)_n], N[(x_i^{L_i} := N_i)_n] \in \mathcal{M}$. Let $x^L \in \mathrm{fv}(M[(x_i^{L_i} := N_i)_n])$ and $x^K \in \mathrm{fv}(N[(x_i^{L_i} := N_i)_n])$. So $x^L \in \mathrm{fv}(M) \cup \mathrm{fv}(N_1) \cup ... \cup \mathrm{fv}(N_n)$ and $x^K \in \mathrm{fv}(N) \cup \mathrm{fv}(N_1) \cup ... \cup \mathrm{fv}(N_n)$. Since $\diamond\mathcal{X}$, then $K = L$. Hence, $M[(x_i^{L_i} := N_i)_n] \diamond N[(x_i^{L_i} := N_i)_n]$ □

Proof (Of Theorem 1).

1. By induction on $M \rhd_\eta^* N$, we only do the base step:
   - $M = \lambda x^L.Nx^L \rhd_\eta N$ and $x^L \notin \mathrm{fv}(N)$. By definition $\mathrm{fv}(M) = \mathrm{fv}(Nx^L) \setminus \{x^L\} = \mathrm{fv}(N)$ and $\mathrm{d}(M) = \mathrm{d}(Nx^L) = \mathrm{d}(N)$.
   - $M = \lambda x^L.M_1 \rhd_\eta \lambda x^L.N_1 = N$ and $M_1 \rhd_\eta N_1$. By IH, $\mathrm{fv}(N_1) = \mathrm{fv}(M_1)$ and $\mathrm{d}(M_1) = \mathrm{d}(N_1)$. Hence, $\mathrm{d}(M) = \mathrm{d}(M_1) = \mathrm{d}(N_1) = \mathrm{d}(N)$ and $\mathrm{fv}(N) = \mathrm{fv}(N_1) \setminus \{x^L\} = \mathrm{fv}(M_1) \setminus \{x^L\} = \mathrm{fv}(M)$.
   - $M = M_1M_2 \rhd_\eta N_1M_2 = N$ such that $M_1 \rhd_\eta N_1$. By IH, $\mathrm{fv}(N_1) = \mathrm{fv}(M_1)$ and $\mathrm{d}(M_1) = \mathrm{d}(N_1)$. By definition, $\mathrm{fv}(N) = \mathrm{fv}(N_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M)$ and $\mathrm{d}(M) = \mathrm{d}(M_1) = \mathrm{d}(N_1) = \mathrm{d}(N)$.
   - $M = M_1M_2 \rhd_\eta M_1N_2 = N$ such that $M_2 \rhd_\eta N_2$. By IH, $\mathrm{fv}(N_2) = \mathrm{fv}(M_2)$ and $\mathrm{d}(M_2) = \mathrm{d}(N_2)$. By definition, $\mathrm{fv}(N) = \mathrm{fv}(M_1) \cup \mathrm{fv}(N_2) = \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M)$ and $\mathrm{d}(M) = \mathrm{d}(M_1) = \mathrm{d}(N)$.
2. Case $r = \beta$. By induction on $M \rhd_\beta^* N$, we only do the base step:
   - $M = (\lambda x^L.M_1)M_2 \rhd_\beta M_1[x^L := M_2] = N$ such that $\mathrm{d}(M_2) = L$. If $x^L \in \mathrm{fv}(M_1)$ then $\mathrm{fv}(N) = (\mathrm{fv}(M_1) \setminus \{x^L\}) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M)$. If $x^L \notin \mathrm{fv}(M_1)$ then $\mathrm{fv}(N) = \mathrm{fv}(M_1) = \mathrm{fv}(M_1) \setminus \{x^L\} \subseteq \mathrm{fv}(M)$. By definition, $\mathrm{d}(M) = \mathrm{d}(M_1)$. Because $N \in \mathcal{M}$ then $M_1 \diamond M_2$ and $\mathrm{d}(M_2) = L$. So, by lemma 18.5, $\mathrm{d}(N) = \mathrm{d}(M_1)$.

- $M = \lambda x^L.M_1 \rhd_\beta \lambda x^L.N_1 = N$ such that $M_1 \rhd_\beta N_1$. By IH, $\mathrm{fv}(N_1) \subseteq \mathrm{fv}(M_1)$ and $\mathrm{d}(M_1) = \mathrm{d}(N_1)$. By definition $\mathrm{d}(M) = \mathrm{d}(M_1) = \mathrm{d}(N_1) = \mathrm{d}(N)$ and $\mathrm{fv}(N) = \mathrm{fv}(N_1) \setminus \{x^L\} \subseteq \mathrm{fv}(M_1) \setminus \{x^L\} = \mathrm{fv}(M)$.
- $M = M_1M_2 \rhd_\beta N_1M_2 = N$ such that $M_1 \rhd_\beta N_1$. By IH, $\mathrm{fv}(N_1) \subseteq \mathrm{fv}(M_1)$ and $\mathrm{d}(M_1) = \mathrm{d}(N_1)$. By definition, $\mathrm{fv}(N) = \mathrm{fv}(N_1) \cup \mathrm{fv}(M_2) \subseteq \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M)$ and $\mathrm{d}(M) = \mathrm{d}(M_1) = \mathrm{d}(N_1) = \mathrm{d}(N)$.
- $M = M_1M_2 \rhd_\beta M_1N_2 = N$ such that $M_2 \rhd_\beta N_2$. By IH, $\mathrm{fv}(N_2) \subseteq \mathrm{fv}(M_2)$ and $\mathrm{d}(M_2) = \mathrm{d}(N_2)$. By definition, $\mathrm{fv}(N) = \mathrm{fv}(M_1) \cup \mathrm{fv}(N_2) \subseteq \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M)$ and $\mathrm{d}(M) = \mathrm{d}(M_1) = \mathrm{d}(N)$.

Case $r = \beta\eta$, by the β and η cases. Case $r = h$, by the β case. $\square$

The next lemma is again needed in the proofs.

Lemma 19. *Let $i, p \geq 0$, $M, N, N_1, N_2, \ldots, N_p \in \mathcal{M}$, $\blacktriangleright' \in \{\rhd^*_\beta, \rhd^*_\eta, \rhd^*_{\beta\eta}\}$ and $\blacktriangleright \in \{\rhd_\beta, \rhd_\eta, \rhd_{\beta\eta}, \rhd_h, \rhd^*_\beta, \rhd^*_\eta, \rhd^*_{\beta\eta}, \rhd^*_h\}$. We have:*

1. *$M^{+i} \in \mathcal{M}$ and $d(M^{+i}) = i :: d(M)$ and x^K occurs in M^{+i} iff $K = i :: L$ and x^L occurs in M.*
2. *$M \diamond N$ iff $M^{+i} \diamond N^{+i}$.*
3. *Let $\mathcal{X} \subseteq \mathcal{M}$ then $\diamond\mathcal{X}$ iff $\diamond\mathcal{X}^{+i}$.*
4. *$(M^{+i})^{-i} = M$.*
5. *If $\diamond\{M\} \cup \{N_j \;/\; j \in \{1, \ldots, p\}\}$ then $(M[(x_j^{L_j} := N_j)_p])^{+i} = M^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$.*
6. *If $M \blacktriangleright N$, then $M^{+i} \blacktriangleright N^{+i}$.*
7. *If $d(M) = i :: L$, then:*
   (a) $M = P^{+i}$ for some $P \in \mathcal{M}$, $d(M^{-i}) = L$ and $(M^{-i})^{+i} = M$.
   (b) If $\forall 1 \leq j \leq p, d(N_j) = i :: K_j$ and $\diamond\{M\} \cup \{N_j \;/\; j \in \{1, \ldots, p\}\}$ then $(M[(x_j^{i::K_j} := N_j)_p])^{-i} = M^{-i}[(x_j^{K_j} := N_j^{-i})_p]$.
   (c) If $M \blacktriangleright N$ then $M^{-i} \blacktriangleright N^{-i}$.
8. *If $M \blacktriangleright N$, $P \blacktriangleright Q$ and $M \diamond P$ then $N \diamond Q$*
9. *If $M \blacktriangleright N^{+i}$, then there is $P \in \mathcal{M}$ such that $M = P^{+i}$ and $P \blacktriangleright N$.*
10. *If $M^{+i} \blacktriangleright N$, then there is $P \in \mathcal{M}$ such that $N = P^{+i}$ and $M \blacktriangleright P$.*
11. *If $y^K \notin \mathrm{fv}(N) \cup \{x^L\}$, $d(P) = K$, $d(N) = L$, $\diamond\{M, N, P\}$ then $M[y^K := P][x^L := N] = M[x^L := N][y^K := P[x^L := N]]$.*
12. *If $M \blacktriangleright N$ and $d(P) = L$ and $\diamond\{M, N, P\}$, then $M[x^L := P] \blacktriangleright N[x^L := P]$.*
13. *If $N \blacktriangleright' P$ and $d(N) = L = d(P)$ and $\diamond\{M, N, P\}$, then $M[x^L := N] \blacktriangleright' M[x^L := P]$.*
14. *If $M \blacktriangleright' M'$, $P \blacktriangleright' P'$ and $d(P) = L$ and $\diamond\{M, M', P, P'\}$, then $M[x^L := P] \blacktriangleright' M'[x^L := P']$.*

Proof. 1 We only prove the lemma by induction on M:
- If $M = x^L$ then $M^{+i} = x^{i::L} \in \mathcal{M}$ and $\mathrm{d}(x^{i::L}) = i :: L = i :: \mathrm{d}(x^L)$.
- If $M = \lambda x^L.M_1$ then $M_1 \in \mathcal{M}$, $L \succeq \mathrm{d}(M_1)$ and $M^{+i} = \lambda x^{i::L}.M_1^{+i}$. By IH, $M_1^{+i} \in \mathcal{M}$ and $\mathrm{d}(M_1^{+i}) = i :: \mathrm{d}(M_1)$ and x^K occurs in M_1^{+i} iff $K = i :: K'$ and $y^{K'}$ occurs in M_1. So $i :: L \succeq i :: \mathrm{d}(M_1) = \mathrm{d}(M_1^{+i})$. Hence, $\lambda x^{i::L}.M_1^{+i} \in \mathcal{M}$. Moreover, $\mathrm{d}(M^{+i}) = \mathrm{d}(M_1^{+i}) = i :: \mathrm{d}(M_1) =$

$i :: \mathrm{d}(M)$. If y^K occurs in M^{+i} then either $y^K = x^{i::L}$, so it is done because x^L occurs in M. Or y^K occurs in M_1^{+i}. By IH, $K = i :: K'$ and $y^{K'}$ occurs in M_1. So $y^{K'}$ occurs in M. If y^K occurs in M then either $y^K = x^L$ and then $y^{i::K}$ occurs in M^{+i}. Or y^K occurs in M_1. Then by IH, $y^{i::K}$ occurs in M_1^{+i}. So, $y^{i::K}$ occurs in M^{+i}.

- If $M = M_1M_2$ then $M_1, M_2 \in \mathcal{M}$, $\mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$, $M_1 \diamond M_2$ and $M^{+i} = M_1^{+i}M_2^{+i}$. By IH, $M_1^{+i}, M_2^{+i} \in \mathcal{M}$, $\mathrm{d}(M_1^{+i}) = i :: \mathrm{d}(M_1)$, $\mathrm{d}(M_2^{+i}) = i :: \mathrm{d}(M_2)$, y^K occurs in M_1^{+i} iff $K = i :: K'$ and $y^{K'}$ occurs in M_1, and y^K occurs in M_2^{+i} iff $K = i :: K'$ and $y^{K'}$ occurs in M_2. Let $x^L \in \mathrm{fv}(M_1^{+i})$ and $x^K \in \mathrm{fv}(M_2^{+i})$ then, using IH, $L = i :: L'$, $K = i :: K'$, $x^{L'}$ occurs in M_1 and $x^{K'}$ occurs in M_2. Using $M_1 \diamond M_2$, we obtain $L' = K'$, so $L = K$. Hence, $M_1^{+i} \diamond M_2^{+i}$. Because $\mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$, then $\mathrm{d}(M_1^{+i}) = i :: \mathrm{d}(M_1) \preceq i :: \mathrm{d}(M_2) = \mathrm{d}(M_2^{+i})$. So, $M^{+i} \in \mathcal{M}$. Moreover, $\mathrm{d}(M^{+1}) = \mathrm{d}(M_1^{+i}) = i :: \mathrm{d}(M_1) = i :: \mathrm{d}(M)$. If x^L occurs in M^{+i} then either x^L occurs in M_1^{+i} and using IH, $L = i :: L'$ and $x^{L'}$ occurs in M_1, so $x^{L'}$ occurs in M. Or x^L occurs in M_2^{+i} and using IH, $L = i :: L'$ and $x^{L'}$ occurs in M_2, so $x^{L'}$ occurs in M. If x^L occurs in M then either x^L occurs in M_1 so by IH $x^{i::L}$ occurs in M_1^{+i}, hence $x^{i::L}$ occurs in M^{+i}. Or x^L occurs in M_2 so by IH $x^{i::L}$ occurs in M_2^{+i}, hence $x^{i::L}$ occurs in M^{+i}.

2 Assume $M \diamond N$. Let $x^L \in \mathrm{fv}(M^{+i})$ and $x^K \in \mathrm{fv}(N^{+i})$ then by lemma 19.1, $L = i :: L'$, $K = i :: K'$, $x^{L'} \in \mathrm{fv}(M)$ and $x^{K'} \in \mathrm{fv}(N)$. Using $M \diamond N$ we obtain $K' = L'$ and so $K = L$.
Assume $M^{+i} \diamond N^{+i}$. Let $x^L \in \mathrm{fv}(M)$ and $x^K \in \mathrm{fv}(N)$, then by lemma 19.1, $x^{i::L} \in \mathrm{fv}(M^{+i})$ and $x^{i::K} \in \mathrm{fv}(N^{+i})$. Using $M^{+i} \diamond N^{+i}$ we obtain $i :: K = i :: L$ and so $K = L$.

3 Let $\mathcal{X} \subseteq \mathcal{M}$.
Assume $\diamond\mathcal{X}$. Let $M, N \in \mathcal{X}^{+i}$. Then by definition, $M = P^{+i}$ and $N = Q^{+i}$ such that $P, Q \in \mathcal{X}$. Because by hypothesis $P \diamond Q$ then by lemma 19.2, $M \diamond N$.
Assume $\diamond\mathcal{X}^{+i}$. Let $M, N \in \mathcal{X}$ then $M^{+i}, N^{+i} \in \mathcal{X}^{+i}$. Because by hypothesis $M^{+i} \diamond N^{+i}$ then by lemma 19.2, $M \diamond N$.

4 By lemma 19.1, $M^{+i} \in \mathcal{M}$ and $\mathrm{d}(M^{+i}) = i :: \mathrm{d}(M)$. We prove the lemma by induction on M.

- Let $M = x^L$ then $M^{+i} = x^{i::L}$ and $(M^{+i})^{-i} = x^L$.
- Let $M = \lambda x^L.M_1$ such that $M_1 \in \mathcal{M}$ and $L \succeq \mathrm{d}(M_1)$. Then, $(M^{+i})^{-i} = (\lambda x^{i::L}.M_1^{+i})^{-i} = \lambda x^L.(M_1^{+i})^{-i} =^{IH} \lambda x^L.M_1$.
- Let $M = M_1M_2$ such that $M_1, M_2 \in \mathcal{M}$, $M_1 \diamond M_2$ and $\mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$. Then, $(M^{+i})^{-i} = (M_1^{+i}M_2^{+i})^{-i} = (M_1^{+i})^{-i}(M_2^{+i})^{-i} =^{IH} M_1M_2$.

5 By 3, $\diamond\{M^{+i}\} \cup \{N_j^{+i} \ / \ j \in \{1, \ldots, p\}\}$. By lemma 18.5, $M[(x_j^{L_j} := N_j)_p]$ and $M^{+i}[(x_j^{i::L_j} := N_j^{+i})_p] \in \mathcal{M}$. By induction on M:

- Let $M = y^K$. If $\forall 1 \leq j \leq p, y^K \neq x_j^{L_j}$ then $y^K[(x_j^{L_j} := N_j)_p] = y^K$. Hence $(y^K[(x_j^{L_j} := N_j)_p])^{+i} = y^{i::K} = y^{i::K}[(x_j^{i::L_j} := N_j^{+i})_p]$. If $\exists 1 \leq j \leq p, y^K = x_j^{L_j}$ then $y^K[(x_j^{L_j} := N_j)_p] = N_j$. Hence $(y^K[(x_j^{L_j} := N_j)_p])^{+i} = N_j^{+i} = y^{i::K}[(x_j^{i::L_j} := N_j^{+i})_p]$.

- Let $M = \lambda y^K.M_1$. Then $M[(x_j^{L_j} := N_j)_p] = \lambda y^K.M_1[(x_j^{L_j} := N_j)_p]$ where $\forall 1 \leq j \leq p, y^K \notin \mathrm{fv}(N_j) \cup \{x_j^{L_j}\}$. By lemma 18.3, $\diamond\{M_1\} \cup \{N_j \ / \ j \in \{1,\dots,p\}\}$. By IH, $(M_1[(x_j^{L_j} := N_j)_p])^{+i} = M_1^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$. Hence, $(M[(x_j^{L_j} := N_j)_p])^{+i} = \lambda y^{i::K}.(M_1[(x_j^{L_j} := N_j)_p])^{+i} =$ $\lambda y^{i::K}.M_1^{+i}[(x_j^{i::L_j} := N_j^{+i})_p] = (\lambda y^K.M_1)^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$.
- Let $M = M_1M_2$. $M[(x_j^{L_j} := N_j)_p] = M_1[(x_j^{L_j} := N_j)_p]M_2[(x_j^{L_j} := N_j)_p]$. By lemma 18.3, $\diamond\{M_1\} \cup \{N_j \ / \ j \in \{1,\dots,p\}\}$ and $\diamond\{M_2\} \cup \{N_j \ / \ j \in \{1,\dots,p\}\}$. By IH, $(M_1[(x_j^{L_j} := N_j)_p])^{+i} = M_1^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$ and $(M_2[(x_j^{L_j} := N_j)_p])^{+i} = M_2^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$. Hence $(M[(x_j^{L_j} := N_j)_p])^{+i} = (M_1[(x_j^{L_j} := N_j)_p])^{+i}(M_2[(x_j^{L_j} := N_j)_p])^{+i} =$ $M_1^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]M_2^{+i}[(x_j^{i::L_j} := N_j^{+i})_p] = M^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$.

6 By lemma 19.1, if $M, N \in \mathcal{M}$ then $M^{+i}, N^{+i} \in \mathcal{M}$.
- Let $\blacktriangleright$ be $\rhd_\beta$. By induction on $M \rhd_\beta N$.
  - Let $M = (\lambda x^L.M_1)M_2 \rhd_\beta M_1[x^L := M_2] = N$ where $\mathrm{d}(M_2) = L$, then by lemma 19.1, $\mathrm{d}(M_2^{+i}) = i :: L$ and $M^{+i} = (\lambda x^{i::L}.M_1^{+i})M_2^{+i} \rhd_\beta M_1^{+i}[x^{i::L} := M_2^{+i}] = (M_1[x^L := M_2])^{+i}$.
  - Let $M = \lambda x^L.M_1 \rhd_\beta \lambda x^L.N_1 = N$ such that $M_1 \rhd_\beta N_1$. By IH, $M_1^{+i} \rhd_\beta N_1^{+i}$, hence $M^{+i} = \lambda x^{i::L}.M_1^{+i} \rhd_\beta \lambda x^{i::L}N_1^{+i} = N^{+i}$.
  - Let $M = M_1M_2 \rhd_\beta N_1M_2 = N$ such that $M_1 \rhd_\beta N_1$. By IH, $M_1^{+i} \rhd_\beta N_1^{+i}$, hence $M^{+i} = M_1^{+i}M_2^{+i} \rhd_\beta N_1^{+i}M_2^{+i} = N^{+i}$.
  - Let $M = M_1M_2 \rhd_\beta M_1N_2 = N$ such that $M_2 \rhd_\beta N_2$. By IH, $M_2^{+i} \rhd_\beta N_2^{+i}$, hence $M^{+i} = M_1^{+i}M_2^{+i} \rhd_\beta N_1^{+i}M_2^{+i} = N^{+i}$.
- Let $\blacktriangleright$ be $\rhd_\beta^*$. By induction on $\rhd_\beta^*$ using $\rhd_\beta$.
- Let $\blacktriangleright$ be $\rhd_\eta$. We only do the base case. The inductive cases are as for $\rhd_\beta$. Let $M = \lambda x^L.Nx^L \rhd_\eta N$ where $x^L \notin \mathrm{fv}(N)$. By lemma 19.1, $x^{i::L} \notin \mathrm{fv}(N^{+i})$ Then $M^{+i} = \lambda x^{i::L}.N^{+i}x^{i::L} \rhd_\eta N^{+i}$.
- Let $\blacktriangleright$ be $\rhd_\eta^*$. By induction on $\rhd_\eta^*$ using $\rhd_\eta$.
- Let $\blacktriangleright$ be $\rhd_{\beta\eta}$, $\rhd_{\beta\eta}^*$, $\rhd_h$ or $\rhd_h^*$. By the previous items.

7 (a) By induction on M:
- Let $M = y^{i::L}$ then $y^L \in \mathcal{M}$ and $\mathrm{d}((y^{i::L})^{-i}) = \mathrm{d}(y^L) = L$ and $((y^{i::L})^{-i})^{+i} = y^{i::L}$.
- Let $M = \lambda y^K.M_1$ such that $M_1 \in \mathcal{M}$ and $K \succeq \mathrm{d}(M_1)$. Because $\mathrm{d}(M_1) = \mathrm{d}(M) = i :: L$, by IH, $M_1 = P^{+i}$ for some $P \in \mathcal{M}$, $\mathrm{d}(M_1^{-i}) = L$ and $(M_1^{-i})^{+i} = M_1$. Because, $K \succeq i :: L$ then $K = i :: L :: K'$ for some K'. Let $Q = \lambda y^{L::K'}.P$. Because $P =^{19.4} (P^{+i})^{-i} = M_1^{-i}$, then $\mathrm{d}(P) = L$. Because $L \preceq L :: K'$, then $Q \in \mathcal{M}$ and $Q^{+i} = M$. Moreover, $\mathrm{d}(M^{-i}) =^{19.4} \mathrm{d}(Q) = \mathrm{d}(P) = L$ and $(M^{-i})^{+i} = P^{+i} = M$.
- Let $M = M_1M_2$ such that $M_1, M_2 \in \mathcal{M}$, $M_1 \diamond M_2$ and $\mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$. Then $\mathrm{d}(M) = \mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$, so $\mathrm{d}(M_2) = i :: L :: L'$ for some L'. By IH $M_1 = P_1^{+i}$ for some $P_1 \in \mathcal{M}$, $\mathrm{d}(M_1^{-i}) = L$ and $(M_1^{-i})^{+i} = M_1$. Again by IH, $M_2 = P_2^{+i}$ for some $P_2 \in \mathcal{M}$, $\mathrm{d}(M_2^{-i}) = L :: L'$ and $(M_2^{-i})^{+i} = M_2$. If $y^{K_1} \in \mathrm{fv}(P_1)$ and $y^{K_2} \in$

$\mathrm{fv}(P_2)$, then by lemma 19.1, $K'_1 = i :: K_1$, $K'_2 = i :: K_2$, $x^{K'_1} \in \mathrm{fv}(M_1)$ and $x^{K'_2} \in \mathrm{fv}(M_2)$. Thus $K'_1 = K'_2$, so $K_1 = K_2$ and $P_1 \diamond P_2$. Because $\mathrm{d}(P_1) = \mathrm{d}(M_1^{-i}) = L \preceq L :: L' = \mathrm{d}(M_2^{-i}) = \mathrm{d}(P_2)$ then $Q = P_1P_2 \in \mathcal{M}$ and $Q^{+i} = (P_1P_2)^{+i} = P_1^{+i}P_2^{+i} = M$. Moreover, $\mathrm{d}(M^{-i}) =^{19.4} \mathrm{d}(Q) = \mathrm{d}(P_1) = L$ and $(M^{-i})^{+i} = Q^{+i} = M$.

(b) By the previous item, there exist $M', N'_1, \dots, N'_n \in \mathcal{M}$ such that $M = M'^{+i}$ and for all $j \in \{1, \dots, p\}$, $N_j = N_j'^{+i}$. So by lemma 19.3, $\diamond\{M'\} \cup \{N'_j \ / \ j \in \{1, \dots, p\}\}$. By lemma 19.4, $M^{-i} = M'$ and for all $j \in \{1, \dots, p\}$, $N_j^{-i} = N'_j$. So, $\diamond\{M^{-i}\} \cup \{N_j^{-i} \ / \ j \in \{1, \dots, p\}\}$. By lemma 18.5, $M[(x_j^{i::K_j} := N_j)_p], M^{-i}[(x_j^{K_j} := N_j^{-i})_p] \in \mathcal{M}$ and $\mathrm{d}(M[(x_j^{i::K_j} := N_j)_p]) = \mathrm{d}(M) = i :: L$. We prove the result by induction on M:

- Let $M = y^{i::L}$. If $\forall 1 \leq j \leq p, y^{i::L} \neq x_j^{i::K_j}$ then $y^{i::L}[(x_j^{i::K_j} := N_j)_p] = y^{i::L}$. Hence $(y^{i::L}[(x_j^{i::K_j} := N_j)_p])^{-i} = y^L = y^L[(x_j^{K_j} := N_j^{-i})_p]$. If $\exists 1 \leq j \leq p, y^{i::L} = x_j^{i::K_j}$ then $y^{i::L}[(x_j^{i::K_j} := N_j)_p] = N_j$. Hence $(y^{i::L}[(x_j^{i::K_j} := N_j)_p])^{-i} = N_j^{-i} = y^L[(x_j^{K_j} := N_j^{-i})_p]$.
- Let $M = \lambda y^K.M_1$ such that $M_1 \in \mathcal{M}$ and $K \succeq \mathrm{d}(M_1)$. Then, $M[(x_j^{i::K_j} := N_j)_p] = \lambda y^K.M_1[(x_j^{i::K_j} := N_j)_p]$ where $\forall 1 \leq j \leq p, y^K \notin \mathrm{fv}(N_j) \cup \{x_j^{i::K_j}\}$. By lemma 18.3, $\diamond\{M_1\} \cup \{N_j \ / \ j \in \{1, \dots, p\}\}$. By definition $\mathrm{d}(M) = \mathrm{d}(M_1)$. By IH, $(M_1[(x_j^{i::K_j} := N_j)_p])^{-i} = M_1^{-i}[(x_j^{K_j} := N_j^{-i})_p]$. Because $\mathrm{d}(M_1) = i :: L \preceq K$, $K = i :: L :: K'$ for some K'.
  Hence, $(M[(x_j^{i::K_j} := N_j)_p])^{-i} = \lambda y^{L::K'}.(M_1[(x_j^{i::K_j} := N_j)_p])^{-i} = \lambda y^{L::K'}.M_1^{-i}[(x_j^{K_j} := N_j^{-i})_p] = (\lambda y^K.M_1)^{-i}[(x_j^{K_j} := N_j^{-i})_p]$.
- Let $M = M_1M_2$ such that $M_1, M_2 \in \mathcal{M}$, $M_1 \diamond M_2$ and $\mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$. Then, $M[(x_j^{i::K_j} := N_j)_p] = M_1[(x_j^{i::K_j} := N_j)_p]M_2[(x_j^{i::K_j} := N_j)_p]$. By lemma 18.3, $\diamond\{M_1\} \cup \{N_j \ / \ j \in \{1, \dots, p\}\}$ and $\diamond\{M_2\} \cup \{N_j \ / \ j \in \{1, \dots, p\}\}$. By definition $\mathrm{d}(M) = \mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$. So $\mathrm{d}(M_2) = i :: L :: L'$ for some L'. By IH, $(M_1[(x_j^{i::K_j} := N_j)_p])^{-i} = M_1^{-i}[(x_j^{K_j} := N_j^{-i})_p]$ and $(M_2[(x_j^{i::K_j} := N_j)_p])^{-i} = M_2^{-i}[(x_j^{K_j} := N_j^{-i})_p]$. Hence
  $(M[(x_j^{i::K_j} := N_j)_p])^{-i} = (M_1[(x_j^{i::K_j} := N_j)_p])^{-i}(M_2[(x_j^{i::K_j} := N_j)_p])^{-i}$
  $= M_1^{-i}[(x_j^{K_j} := N_j^{-i})_p]M_2^{-i}[(x_j^{K_j} := N_j^{-i})_p] = M^{-i}[(x_j^{K_j} := N_j^{-i})_p]$.

(c) Using lemma 19.4, lemma 1 and the first item, we prove that $M^{-i}, N^{-i} \in \mathcal{M}$.

- Let $\blacktriangleright$ be $\rhd_\beta$. By induction on $M \rhd_\beta N$.
  - Let $M = (\lambda x^K.M_1)M_2 \rhd_\beta M_1[x^K := M_2] = N$ where $\mathrm{d}(M_2) = K$. Because $M \in \mathcal{M}$ then $M_1 \in \mathcal{M}$. Because $i :: L = \mathrm{d}(M) = \mathrm{d}(M_1) \preceq K$, then $K = i :: L :: K'$. By lemma 19.7, $\mathrm{d}(M_2^{-i}) = L :: K'$. So $M^{-i} = (\lambda x^{L::K'}.M_1^{-i})M_2^{-i} \rhd_\beta M_1^{-i}[x^{L::K'} := M_2^{-i}] = (M_1[x^K := M_2])^{-i}$.

  - Let $M = \lambda x^K.M_1 \rhd_\beta \lambda x^K.N_1 = N$ such that $M_1 \rhd_\beta N_1$. Because $M \in \mathcal{M}$, $M_1 \in \mathcal{M}$ and $K \succeq \mathrm{d}(M_1)$. By definition $\mathrm{d}(M) = \mathrm{d}(M_1)$. Because $i :: L = \mathrm{d}(M_1) \preceq K$, $K = i :: L :: K'$ for some K'. By IH, $M_1^{-i} \rhd_\beta N_1^{-i}$, hence $M^{-i} = \lambda x^{L::K'}.M_1^{-i} \rhd_\beta \lambda x^{L::K'} N_1^{-i} = N^{-i}$.
  - Let $M = M_1M_2 \rhd_\beta N_1M_2 = N$ such that $M_1 \rhd_\beta N_1$. Because $M \in \mathcal{M}$ then $M_1 \in \mathcal{M}$. By definition $\mathrm{d}(M) = \mathrm{d}(M_1) = i :: L$. By IH, $M_1^{-i} \rhd_\beta N_1^{-i}$, hence $M^{-i} = M_1^{-i}M_2^{-i} \rhd_\beta N_1^{-i}M_2^{-i} = N^{-i}$.
  - Let $M = M_1M_2 \rhd_\beta M_1N_2 = N$ such that $M_2 \rhd_\beta N_2$. Because $M \in \mathcal{M}$ then $M_2 \in \mathcal{M}$. By definition $\mathrm{d}(M_2) \succeq \mathrm{d}(M_1) = \mathrm{d}(M) = i :: L$. So $\mathrm{d}(M_2) = i :: L :: L'$ for some L'. By IH, $M_2^{-i} \rhd_\beta N_2^{-i}$, hence $M^{-i} = M_1^{-i}M_2^{-i} \rhd_\beta N_1^{-i}M_2^{-i} = N^{-i}$.
- Let $\blacktriangleright$ be $\rhd_\beta^*$. By induction on $\rhd_\beta^*$. using $\rhd_\beta$.
- Let $\blacktriangleright$ be $\rhd_\eta$. We only do the base case. The inductive cases are as for $\rhd_\beta$. Let $M = \lambda x^K.Nx^K \rhd_\eta N$ where $x^K \notin \mathrm{fv}(N)$. Because $i :: L = \mathrm{d}(M) = \mathrm{d}(N) \preceq K$, then $K = i :: L :: K'$ for some K'. By lemma 19.7, $N = N'^{+i}$ for some $N' \in \mathcal{M}$. By lemma 19.7, $N' = N^{-i}$. By lemma 19.1, $x^{L::K'} \notin \mathrm{fv}(N^{-i})$. Then $M^{-i} = \lambda x^{L::K'}.N^{-i}x^{L::K'} \rhd_\eta N^{-i}$.
- Let $\blacktriangleright$ be $\rhd_\eta^*$. By induction on $\rhd_\eta^*$ using $\rhd_\eta$.
- Let $\blacktriangleright$ be $\rhd_{\beta\eta}$, $\rhd_{\beta\eta}^*$, $\rhd_h$ or $\rhd_h^*$. By the previous items.

8 Let $x^L \in \mathrm{fv}(N) \subseteq^1 \mathrm{fv}(M)$ and $X^K \in \mathrm{fv}(Q) \subseteq^1 \mathrm{fv}(P)$, since $M \diamond P$, $L = K$. Hence $N \diamond Q$.

9 By lemma 19.1, $\mathrm{d}(N^{+i}) = i :: \mathrm{d}(N)$. By lemma 1, $\mathrm{d}(M) = \mathrm{d}(N^{+i})$. By lemma 19.7, $M = M'^{+i}$ such that $M' \in \mathcal{M}$. By lemma 19.7, $M' =^{19.4} (M'^{+i})^{-i} = M^{-i} \blacktriangleright (N^{+i})^{-i} =^{19.4} N$.

10 By lemma 19.1, $\mathrm{d}(M^{+i}) = i :: \mathrm{d}(M)$. By lemma 1, $\mathrm{d}(M^{+i}) = \mathrm{d}(N)$. By lemma 19.7, $N = N'^{+i}$ such that $N' \in \mathcal{M}$. By lemma 19.7, $M =^{19.4} (M^{+i})^{-i} \blacktriangleright N^{-i} = (N'^{+i})^{-i} =^{19.4} N'$.

11 By lemma 18.5, $M[y^K := P] \in \mathcal{M}$. Let us now prove $\diamond\{M[y^K := P], N\}$. Let $z^R \in \mathrm{fv}(M[y^K := P])$ and $z^{R'} \in \mathrm{fv}(N)$ then $z^R \in \mathrm{fv}(M)$ or $z^R \in \mathrm{fv}(P)$. In both cases, because $M \diamond N$ and $P \diamond N$, we obtain $R = R'$. So by lemma 18.5, $M[y^K := P][x^L := N] \in \mathcal{M}$.
By lemma 18.5, $M[x^L := N], P[x^L := N] \in \mathcal{M}$ and $\mathrm{d}(P[x^L := N]) = \mathrm{d}(P) = K$. Let us now prove that $\diamond\{M[x^L := N], P[x^L := N]\}$. Let $z^R \in \mathrm{fv}(M[x^L := N)$ and $z^{R'} \in \mathrm{fv}(P[x^L := N])$ then either $z^R \in \mathrm{fv}(M)$ or $z^R \in \mathrm{fv}(N)$ and either $z^{R'} \in \mathrm{fv}(P)$ or $z^{R'} \in \mathrm{fv}(N)$. In all of the four cases, because by hypotheses and lemma 18.1, $M \diamond P$, $M \diamond N$, $N \diamond P$ and $N \diamond N$, we obtain $R = R'$. So by lemma 18.5, $M[x^L := N][y^K := P[x^L := N]] \in \mathcal{M}$.
We prove this lemma by induction on the structure of M.
- Let $M = z^R$.
  - If $z^R = y^K$ then $M[y^K := P][x^L := N] = P[x^L := N] = M[y^K := P[x^L := N]] = M[x^L := N][y^K := P[x^L := N]]$.
  - Else
    * If $M = x^L$ then $M[y^K := P][x^L := N] = M[x^L := N] = N = N[y^K := P[x^L := N]] = M[x^L := N][y^K := P[x^L := N]]$.

- ∗ Else $M[y^K := P][x^L := N] = M[x^L := N] = M = M[y^K := P[x^L := N]] = M[x^L := N][y^K := P[x^L := N]]$.
- Let $M = \lambda z^R M_1$ such that $R \succeq \mathrm{d}(M_1)$ and $M_1 \in \mathcal{M}$. By lemma 18.3, $\diamond\{M_1, N, P\}$. Then, $M[y^K := P][x^L := N] = \lambda z^R.M_1[y^K := P][x^L := N] =^{IH} \lambda z^R.M_1[x^L := N][y^K := P[x^L := N]] = M[x^L := N][y^K := P[x^L := N]]$ such that $z^R \notin \mathrm{fv}(N) \cup \mathrm{fv}(P) \cup \{y^K, x^L\}$.
- Let $M = M_1 M_2$ such that $M_1, M_2 \in \mathcal{M}$, $\mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$ and $M_1 \diamond M_2$. By lemma 18.3, $\diamond\{M_1, N, P\}$ and $\diamond\{M_2, N, P\}$. Then, $M[y^K := P][x^L := N] = M_1[y^K := P][x^L := N]M_2[y^K := P][x^L := N] =^{IH} M_1[x^L := N][y^K := P[x^L := N]]M_2[x^L := N][y^K := P[x^L := N]] = M[x^L := N][y^K := P[x^L := N]]$.

12 By lemma 18.5 and using the hypothesis, we obtain $M[x^L := P], N[x^L := P] \in \mathcal{M}$.
- Let $\blacktriangleright = \rhd_\beta$. We prove the result by induction on $M \rhd_\beta N$.
  - Let $M = (\lambda y^K.M_1)M_2 \rhd_\beta M_1[y^K := M_2] = N$ such that $\mathrm{d}(M_2) = K$. Then $M[x^L := P] = (\lambda y^K.M_1[x^L := P])M_2[x^L := P]$ and $N[x^L := P] =^{19.11} M_1[x^L := P][y^K := M_2[x^L := P]]$ such that $y^K \notin \mathrm{fv}(P) \cup \{x^L\}$. By lemma 18.5, $\mathrm{d}(M_2[x^L := P]) = \mathrm{d}(M_2) = K$. So, $M[x^L := P] \rhd_\beta N[x^L := P]$.
  - Let $M = \lambda y^K.M_1 \rhd_\beta \lambda y^K.N_1 = N$ such that $M_1 \rhd_\beta N_1$. Then $M[x^L := P] = \lambda y^K.M_1[x^L := P]$ and $N[x^L := P] = \lambda y^K.N_1[x^L := P]$ such that $y^K \notin \mathrm{fv}(P) \cup \{x^L\}$. By lemma 18.3, $\diamond\{M_1, N_1, P\}$. By IH, $M_1[x^L := P] \rhd_\beta N_1[x^L := P]$. So, $M[x^L := P] \rhd_\beta N[x^L := P]$.
  - Let $M = M_1 M_2 \rhd_\beta N_1 M_2 = N$ such that $M_1 \rhd_\beta N_1$. By lemma 18.3, $\diamond\{M_1, N_1, P\}$. By IH, $M_1[x^L := P] \rhd_\beta N_1[x^L := P]$. So, $M[x^L := P] \rhd_\beta N[x^L := P]$.
  - Let $M = M_1 M_2 \rhd_\beta M_1 N_2 = N$ such that $M_2 \rhd_\beta N_2$. By lemma 18.3, $\diamond\{M_2, N_2, P\}$. By IH, $M_2[x^L := P] \rhd_\beta N_2[x^L := P]$. So, $M[x^L := P] \rhd_\beta N[x^L := P]$.
- Let $\blacktriangleright = \rhd_\eta$. We only prove the base case. The other cases are similar as the ones for $\rhd_\beta$. Let $M = \lambda y^K.N y^K \rhd_\eta N$ such that $y^K \notin \mathrm{fv}(N)$. Then $M[x^L := P] = \lambda y^K.N[x^L := P]y^K$ such that $y^K \notin \mathrm{fv}(P) \cup \{x^L\}$. So $y^K \notin \mathrm{fv}(N[x^L := P])$. Hence, $M[x^L := P] \rhd_\eta N[x^L := P]$.
- The other cases are based on the two previous ones.

13 By lemma 18.5 and using the hypothesis, we obtain $M[x^L := P], M[x^L := N] \in \mathcal{M}$. We prove the result by induction on the structure of M.
- Let $M = y^K$.
  - If $y^K = x^L$ then $M[x^L := P] = P \blacktriangleright' N = M[x^L := N]$.
  - Else, $M[x^L := P] = M \blacktriangleright' M = M[x^L := N]$.
- Let $M = \lambda y^K.M_1$ such that $K \succeq \mathrm{d}(M_1)$ and $M_1 \in \mathcal{M}$. Then $M[x^L := P] = \lambda y^K.M_1[x^L := P]$ and $M[x^L := N] = \lambda y^K.M_1[x^L := N]$ such that $y^K \notin \mathrm{fv}(P) \cup \mathrm{fv}(N) \cup \{x^L\}$. By lemma 18.3, $\diamond\{M_1, N, P\}$. By IH, $M_1[x^L := N] \blacktriangleright' M_1[x^L := P]$. So, $M[x^L := N] \blacktriangleright' M[x^L := P]$.
- Let $M = M_1 M_2$ such that $M_1, M_2 \in \mathcal{M}$, $M_1 \diamond M_2$ and $\mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$. By lemma 18.3, $\diamond\{M_1, N, P\}$ and $\diamond\{M_2, N, P\}$. By IH, $M_1[x^L := N] \blacktriangleright' M_1[x^L := P]$ and $M_2[x^L := N] \blacktriangleright' M_2[x^L := P]$. By lemma 18.5,

$M_1[x^L := N], M_2[x^L := N], M_1[x^L := P], M_2[x^L := P] \in \mathcal{M}$ and $\mathrm{d}(M_1[x^L := N]) = \mathrm{d}(M_1) \preceq \mathrm{d}(M_2) = \mathrm{d}(M_2[x^L := N])$ and $\mathrm{d}(M_1[x^L := P]) = \mathrm{d}(M_1) \preceq \mathrm{d}(M_2) = \mathrm{d}(M_2[x^L := N])$ and $\mathrm{d}(M_1[x^L := P]) = \mathrm{d}(M_1) \preceq \mathrm{d}(M_2) = \mathrm{d}(M_2[x^L := P])$. By lemma 18.6, $M_1[x^L := N] \diamond M_2[x^L := N]$ and $M_1[x^L := P] \diamond M_2[x^L := N]$ and $M_1[x^L := P] \diamond M_2[x^L := P]$. So $M_1[x^L := N]M_2[x^L := N], M_1[x^L := P]M_2[x^L := N], M_1[x^L := P]M_2[x^L := P] \in \mathcal{M}$. So $M_1[x^L := N]M_2[x^L := N] \blacktriangleright' M_1[x^L := P]M_2[x^L := N]$ and $M_1[x^L := P]M_2[x^L := N] \blacktriangleright' M_1[x^L := P]M_2[x^L := P]$. Hence, $M[x^L := N] \blacktriangleright' M[x^L := P]$.

14 By lemma 19.12, $M[x^L := P] \blacktriangleright' M'[x^L := P]$. By lemma 19.13, $M'[x^L := P] \blacktriangleright' M'[x^L := P']$. So, $M[x^L := P] \blacktriangleright' M'[x^L := P']$. □

Next we give a lemma that will be used in the rest of the article.

Lemma 20. *1. If $M[y^L := x^L] \rhd_\beta N$ then $M \rhd_\beta N'$ where $N = N'[y^L := x^L]$.*
2. If $M[y^L := x^L]$ is β-normalising then M is β-normalising.
3. Let $k \geq 1$. If $Mx_1^{L_1}...x_k^{L_k}$ is β-normalising, then M is β-normalising.
4. Let $k \geq 1$, $1 \leq i \leq k$, $l \geq 0$, $x_i^{L_i}N_1...N_l$ be in normal form and M be closed. If $Mx_1^{L_1}...x_k^{L_k} \rhd_\beta^* x_i^{L_i}N_1...N_l$, then for some $m \geq i$ and $n \leq l$, $M \rhd_\beta^* \lambda x_1^{L_1}....\lambda x_m^{L_m}.x_i^{L_i}M_1...M_n$ where $n+k = m+l$, $M_j \simeq_\beta N_j$ for every $1 \leq j \leq n$ and $N_{n+j} \simeq_\beta x_{m+j}^{L_{m+j}}$ for every $1 \leq j \leq k-m$.

Proof. 1. By induction on $M[y^L := x^L] \rhd_\beta N$.
2. Immediate by 1.
3. By induction on $k \geq 1$. We only prove the basic case. The proof is by cases.
   - If $M\,x_1^{L_1} \rhd_\beta^* M'\,x_1^{L_1}$ where $M'\,x_1^{L_1}$ is in β-normal form and $M \rhd_\beta^* M'$ then M' is in β-normal form and M is β-normalising.
   - If $M\,x_1^{L_1} \rhd_\beta^* (\lambda y^{L_1}.N)\,x_1^{L_1} \rhd_\beta N[y^{L_1} := x_1^{L_1}] \rhd_\beta^* P$ where P is in β-normal form and $M \rhd_\beta^* \lambda y^{L_1}.N$ then by 2, N has a β-normal form and so, $\lambda y^{L_1}.N$ has a β-normal form. Hence, M has a β-normal form.
4. By 3, M is β-normalising and, since M is closed, its β-normal form is $\lambda x_1^{L_1}....\lambda x_m^{L_m}.x_p^{L_p}M_1...M_n$ for $n, m \geq 0$ and $1 \leq p \leq m$.
   Since by theorem 2, $x_i^{L_i}N_1...N_l \simeq_\beta (\lambda x_1^{L_1}....\lambda x_m^{L_m}.x_p^{L_p}M_1...M_n)x_1^{L_1}...x_k^{L_k}$ then $m \leq k$, $x_i^{L_i}\ N_1...N_l \simeq_\beta x_p^{L_p}M_1...M_nx_{m+1}^{L_{m+1}}...x_k^{L_k}$. Hence, $n \leq l$, $i = p \leq m$, $l = n + k - m$, for every $1 \leq j \leq n$, $M_j \simeq_\beta N_j$ and for every $1 \leq j \leq k - m$, $N_{n+j} \simeq_\beta x_{m+j}^{n_{m+j}}$. □

A.1 Confluence of $\rhd_\beta^*$, $\rhd_h^*$ and $\rhd_{\beta\eta}^*$

In this section we establish the confluence of $\rhd_\beta^*$, $\rhd_h^*$ and $\rhd_{\beta\eta}^*$ using the standard parallel reduction method for $\rhd_\beta^*$ and $\rhd_{\beta\eta}^*$.

Definition 17. *Let $r \in \{\beta, \beta\eta\}$. We define on $\mathcal{M}$ the binary relation $\overset{\rho_r}{\to}$ by:*

- *$M \overset{\rho_r}{\to} M$*
- *If $M \overset{\rho_r}{\to} M'$ then $\lambda x^L.M \overset{\rho_r}{\to} \lambda x^L.M'$.*

– *If* $M \stackrel{\rho_r}{\to} M'$*,* $N \stackrel{\rho_r}{\to} N'$ *and* $M \diamond N$ *and* $d(M) \succeq d(N)$ *then* $MN \stackrel{\rho_r}{\to} M'N'$
– *If* $M \stackrel{\rho_r}{\to} M'$*,* $N \stackrel{\rho_r}{\to} N'$*,* $d(N) = L \succeq d(M)$ *and* $M \diamond N$*, then* $(\lambda x^L.M)N \stackrel{\rho_r}{\to} M'[x^n := N']$
– *If* $M \stackrel{\rho_{\beta\eta}}{\to} M'$*,* $x^L \diamond M$ *and* $L \succeq d(M)$ *then* $\lambda x^L.Mx^L \stackrel{\rho_{\beta\eta}}{\to} M'$

We denote the transitive closure of $\stackrel{\rho_r}{\to}$ *by* $\stackrel{\rho_r}{\twoheadrightarrow}$*. When* $M \stackrel{\rho_r}{\to} N$ *(resp.* $M \stackrel{\rho_r}{\twoheadrightarrow} N$*), we can also write* $N \stackrel{\rho_r}{\leftarrow} M$ *(resp.* $N \stackrel{\rho_r}{\twoheadleftarrow} M$*). If* $R, R' \in \{\stackrel{\rho_r}{\to}, \stackrel{\rho_r}{\twoheadrightarrow}, \stackrel{\rho_r}{\leftarrow}, \stackrel{\rho_r}{\twoheadleftarrow}\}$*, we write* $M_1\ R\ M_2\ R'\ M_3$ *instead of* $M_1\ R\ M_2$ *and* $M_2\ R'\ M_3$*.*

Lemma 21. *Let* $M \in \mathcal{M}$*.*

1. *If* $M \rhd_r M'$*, then* $M \stackrel{\rho_r}{\to} M'$*.*
2. *If* $M \stackrel{\rho_r}{\to} M'$*, then* $M' \in \mathcal{M}$*,* $M \rhd_r^* M'$*,* $\mathrm{fv}(M') \subseteq \mathrm{fv}(M)$ *and* $d(M) = d(M')$*.*
3. *If* $M \stackrel{\rho_r}{\to} M'$*,* $N \stackrel{\rho_r}{\to} N'$ *and* $M \diamond N$ *then* $M' \diamond N'$

Proof. 1. By induction on the derivation $M \rhd_r M'$. 2. By induction on the derivation of $M \stackrel{\rho_r}{\to} M'$ using theorem 1 and lemma 19. 3. Let $x^L \in \mathrm{fv}(M')$ and $x^K \in \mathrm{fv}(N')$. By 2., $\mathrm{fv}(M') \subseteq \mathrm{fv}(M)$ and $\mathrm{fv}(N') \subseteq \mathrm{fv}(N)$. Hence, since $M \diamond N$, $L = K$, so $M' \diamond N'$. □

Lemma 22. *Let* $M, N \in \mathcal{M}$*,* $M \diamond N$ *and* $N \stackrel{\rho_r}{\to} N'$*. We have:*

1. $M[x^L := N] \stackrel{\rho_r}{\to} M[x^L := N']$*.*
2. *If* $M \stackrel{\rho_r}{\to} M'$ *and* $d(N) = L$*, then* $M[x^L := N] \stackrel{\rho_r}{\to} M'[x^L := N']$*.*

Proof. 1. By induction on M:

– Let $M = y^K$. If $y^K = x^L$, then $M[x^L := N] = N$, $M[x^L := N'] = N'$ and by hypothesis, $N \stackrel{\rho_r}{\to} N'$. If $y^K \neq x^L$, then $M[x^L := N] = M$, $M[x^L := N'] = M$ and by definition, $M \stackrel{\rho_r}{\to} M$.
– Let $M = \lambda y^K.M_1$. $M[x^L := N] = \lambda y^K.M_1[x^L := N]$ and since $M_1 \diamond N$, by IH, $M_1[x^L := N] \stackrel{\rho_r}{\to} M_1[x^L := N']$ and so $\lambda y^K.M_1[x^L := N] \stackrel{\rho_r}{\to} \lambda y^K.M_1[x^L := N']$
– Let $M = M_1M_2$. $M[x^L := N] = M_1[x^L := N]M_2[x^L := N]$ and since $M_1 \diamond N$ and $M_2 \diamond N$, by IH, $M_1[x^L := N] \stackrel{\rho_r}{\to} M_1[x^L := N']$ and $M_2[x^L := N] \stackrel{\rho_r}{\to} M_2[x^L := N']$. By lemma 18.6, $M_1[x^L := N] \diamond M_2[x^L := N]$, so $M_1[x^L := N]M_2[x^L := N] \stackrel{\rho_r}{\to} M_1[x^L := N']M_2[x^L := N']$.

2. By induction on $M \stackrel{\rho_r}{\to} M'$.

– If $M = M'$, then 1..
– If $\lambda y^K.M \stackrel{\rho_r}{\to} \lambda y^K.M'$ where $M \stackrel{\rho_r}{\to} M'$, then by IH, $M[x^L := N] \stackrel{\rho_r}{\to} M'[x^L := N']$. Hence $(\lambda y^K.M)[x^L := N] = \lambda y^K.M[x^L := N] \stackrel{\rho_r}{\to} \lambda y^K.M'[x^L := N'] = (\lambda y^K.M')[x^L := N']$ where $y^K \notin \mathrm{fv}(N') \subseteq \mathrm{fv}(N)$.
– If $PQ \stackrel{\rho_r}{\to} P'Q'$ where $P \stackrel{\rho_r}{\to} P'$, $Q \stackrel{\rho_r}{\to} Q'$ and $P \diamond Q$, then by IH, $P[x^L := N] \stackrel{\rho_r}{\to} P'[x^L := N']$ and $Q[x^L := N] \stackrel{\rho_r}{\to} Q'[x^L := N']$. By lemma 18.6, $P[x^L := N] \diamond Q[x^L := N]$, so $P[x^L := N]Q[x^L := N] \stackrel{\rho_r}{\to} P'[x^L := N']Q'[x^L := N']$.

– $(\lambda y^K.P)Q \stackrel{\rho_r}{\to} P'[y^K := Q']$ where $P \stackrel{\rho_r}{\to} P'$, $Q \stackrel{\rho_r}{\to} Q'$, $P \diamond Q$ and $\mathrm{d}(Q) = K$, then by IH, $P[x^L := N] \stackrel{\rho_r}{\to} P'[x^L := N']$, $Q[x^L := N] \stackrel{\rho_r}{\to} Q'[x^L := N']$. Moreover, $((\lambda y^K.P)Q)[x^L := N] = (\lambda y^K.P)[x^L := N]Q[x^L := N] = \lambda y^K.P[x^L := N]Q[x^L := N]$ where $y^K \notin \mathrm{fv}(N') \subseteq \mathrm{fv}(N)$. By lemma 18.6, $P[x^L := N] \diamond Q[x^L := N]$ and by lemma 18.5 $\mathrm{d}(Q) = \mathrm{d}(Q[x^L := N])$ so $\lambda y^K.P[x^L := N]Q[x^L := N] \stackrel{\rho_r}{\to} P'[x^L := N'][y^K := Q'[x^L := N']] = P'[y^K := Q'][x^L := N']$.
– If $\lambda y^K.My^K \stackrel{\rho_{\beta\eta}}{\to} M'$ where $M \stackrel{\rho_{\beta\eta}}{\to} M'$, $K \succeq \mathrm{d}(M)$ and $\forall K \in \mathcal{L}_{\mathbb{N}}, y^K \notin \mathrm{fv}(M)$, then by IH $M[x^L := N] \stackrel{\rho_{\beta\eta}}{\to} M'[x^L := N']$. Moreover, $(\lambda y^K.My^K)[x^L := N] = \lambda y^K.M[x^L := N]y^K[x^L := N] = \lambda y^K.M[x^L := N]y^K$ where $\forall K \in \mathcal{L}_{\mathbb{N}}, y^K \notin \mathrm{fv}(N') \subseteq \mathrm{fv}(N)$. Since by lemma 18.5 $\mathrm{d}(M) = \mathrm{d}(M[x^L := N])$, $\lambda y^K.M[x^L := N]y^K \stackrel{\rho_{\beta\eta}}{\to} M'[x^L := N']$. □

Lemma 23. *1. If $x^L \stackrel{\rho_r}{\to} N$, then $N = x^L$.*
2. If $\lambda x^L.P \stackrel{\rho_{\beta\eta}}{\to} N$ then one of the following holds:
  – $N = \lambda x^L.P'$ where $P \stackrel{\rho_{\beta\eta}}{\to} P'$.
  – $P = P'x^L$ where $\forall L \in \mathcal{L}_{\mathbb{N}}, x^L \notin \mathrm{fv}(P')$, $L \succeq d(P')$ and $P' \stackrel{\rho_{\beta\eta}}{\to} N$.
3. If $\lambda x^L.P \stackrel{\rho_\beta}{\to} N$ then $N = \lambda x^L.P'$ where $P \stackrel{\rho_\beta}{\to} P'$.
4. If $PQ \stackrel{\rho_r}{\to} N$, then one of the following holds:
  – $N = P'Q'$, $P \stackrel{\rho_r}{\to} P'$, $Q \stackrel{\rho_r}{\to} Q'$ and $P \diamond Q$.
  – $P = \lambda x^L.P'$, $N = P''[x^L := Q']$, $P' \stackrel{\rho_r}{\to} P''$, $Q \stackrel{\rho_r}{\to} Q'$, $P' \diamond Q$ and $d(Q) = L$.

Proof. 1. By induction on the derivation $x^L \stackrel{\rho_r}{\to} N$.
2. By induction on the derivation $\lambda x^L.P \stackrel{\rho_{\beta\eta}}{\to} N$.
3. By induction on the derivation $\lambda x^L.P \stackrel{\rho_\beta}{\to} N$.
4. By induction on the derivation $PQ \stackrel{\rho_r}{\to} N$. □

Lemma 24. *Let $M, M_1, M_2 \in \mathcal{M}$.*

1. *If $M_2 \stackrel{\rho_r}{\leftarrow} M \stackrel{\rho_r}{\to} M_1$, then there is $M' \in \mathcal{M}$ such that $M_2 \stackrel{\rho_r}{\to} M' \stackrel{\rho_r}{\leftarrow} M_1$.*
2. *If $M_2 \stackrel{\rho_r}{\twoheadleftarrow} M \stackrel{\rho_r}{\twoheadrightarrow} M_1$, then there is $M' \in \mathcal{M}$ such that $M_2 \stackrel{\rho_r}{\twoheadrightarrow} M' \stackrel{\rho_r}{\twoheadleftarrow} M_1$.*

Proof. 1. By induction on M:

– Let $r = \beta\eta$:
  - If $M = x^L$, by lemma 23, $M_1 = M_2 = x^L$. Take $M' = x^L$.
  - If $N_2P_2 \stackrel{\rho_{\beta\eta}}{\leftarrow} NP \stackrel{\rho_{\beta\eta}}{\to} N_1P_1$ where $N_2 \stackrel{\rho_{\beta\eta}}{\leftarrow} N \stackrel{\rho_{\beta\eta}}{\to} N_1$, $P_2 \stackrel{\rho_{\beta\eta}}{\leftarrow} P \stackrel{\rho_{\beta\eta}}{\to} P_1$ and $N \diamond P$ then, by IH, $\exists N', P'$ such that $N_2 \stackrel{\rho_{\beta\eta}}{\to} N' \stackrel{\rho_{\beta\eta}}{\leftarrow} N_1$ and $P_2 \stackrel{\rho_{\beta\eta}}{\to} P' \stackrel{\rho_{\beta\eta}}{\leftarrow} P_1$. By lemma 21.3, $N_1 \diamond P_1$ and $N_2 \diamond P_2$, hence $N_2P_2 \stackrel{\rho_{\beta\eta}}{\to} N'P' \stackrel{\rho_{\beta\eta}}{\leftarrow} N_1P_1$.
  - If $(\lambda x^L.P_1)Q_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} (\lambda x^L.P)Q \stackrel{\rho_{\beta\eta}}{\to} P_2[x^L := Q_2]$ where $\lambda x^L.P \stackrel{\rho_{\beta\eta}}{\to} \lambda x^L.P_1$, $P \stackrel{\rho_{\beta\eta}}{\to} P_2$, $Q_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} Q \stackrel{\rho_{\beta\eta}}{\to} Q_2$, $\mathrm{d}(Q) = L$, $(\lambda x^L.P) \diamond Q$ and $P \diamond Q$ then, by lemma 23, $P \stackrel{\rho_{\beta\eta}}{\to} P_1$. By IH, $\exists P', Q'$ such that $P_1 \stackrel{\rho_{\beta\eta}}{\to} P' \stackrel{\rho_{\beta\eta}}{\leftarrow} P_2$ and $Q_1 \stackrel{\rho_{\beta\eta}}{\to} Q' \stackrel{\rho_{\beta\eta}}{\leftarrow} Q_2$. By lemma 21.2, $\mathrm{d}(Q_1) = \mathrm{d}(Q_2) = \mathrm{d}(Q) = L$. By lemma 21.3, $P_1 \diamond Q_1$. Hence, $(\lambda x^L.P_1)Q_1 \stackrel{\rho_{\beta\eta}}{\to} P'[x^L := Q']$.

Moreover, since $P_2 \stackrel{\rho\beta\eta}{\to} P'$, $Q_2 \stackrel{\rho\beta\eta}{\to} Q'$, $\mathrm{d}(Q_2) = L$ and by lemma 21.3, $P_2 \diamond Q_2$, then, by lemma 22.2, $P_2[x^L := Q_2] \stackrel{\rho\beta\eta}{\to} P'[x^L := Q']$.

- If $P_1[x^L := Q_1] \stackrel{\rho\beta\eta}{\leftarrow} (\lambda x^L.P)Q \stackrel{\rho\beta\eta}{\to} P_2[x^L := Q_2]$ where $P_1 \stackrel{\rho\beta\eta}{\leftarrow} P \stackrel{\rho\beta\eta}{\to} P_2$, $Q_1 \stackrel{\rho\beta\eta}{\leftarrow} Q \stackrel{\rho\beta\eta}{\to} Q_2$, $\mathrm{d}(Q) = L$ and $P \diamond Q$, then, by IH, $\exists P', Q'$ where $P_1 \stackrel{\rho\beta\eta}{\to} P' \stackrel{\rho\beta\eta}{\leftarrow} P_2$ and $Q_1 \stackrel{\rho\beta\eta}{\to} Q' \stackrel{\rho\beta\eta}{\leftarrow} Q_2$. By lemma 21.2, $\mathrm{d}(Q_1) = \mathrm{d}(Q_2) = \mathrm{d}(Q) = L$. By lemma 21.3, $P_1 \diamond Q_1$ and $P_2 \diamond Q_2$. Hence, by lemma 22.2, $P_1[x^L := Q_1] \stackrel{\rho\beta\eta}{\to} P'[x^L := Q'] \stackrel{\rho\beta\eta}{\leftarrow} P_2[x^L := Q_2]$.
- If $\lambda x^L.N_2 \stackrel{\rho\beta\eta}{\leftarrow} \lambda x^L.N \stackrel{\rho\beta\eta}{\to} \lambda x^L.N_1$ where $N_2 \stackrel{\rho\beta\eta}{\leftarrow} N \stackrel{\rho\beta\eta}{\to} N_1$, by IH, there is N' such that $N_2 \stackrel{\rho\beta\eta}{\to} N' \stackrel{\rho\beta\eta}{\leftarrow} N_1$. Hence, $\lambda x^L.N_2 \stackrel{\rho\beta\eta}{\to} \lambda x^L.N' \stackrel{\rho\beta\eta}{\leftarrow} \lambda x^L.N_1$.
- If $M_1 \stackrel{\rho\beta\eta}{\leftarrow} \lambda x^L.Px^L \stackrel{\rho\beta\eta}{\to} M_2$ where $\forall L \in \mathcal{L}_{\mathbb{N}}, x^L \notin \mathrm{fv}(P)$, $L \succeq \mathrm{d}(P)$ and $M_1 \stackrel{\rho\beta\eta}{\leftarrow} P \stackrel{\rho\beta\eta}{\to} M_2$, then, by IH, there is M' such that $M_2 \stackrel{\rho\beta\eta}{\to} M' \stackrel{\rho\beta\eta}{\leftarrow} M_1$.
- If $M_1 \stackrel{\rho\beta\eta}{\leftarrow} \lambda x^L.Px^L \stackrel{\rho\beta\eta}{\to} \lambda x^L.P'$, where $P \stackrel{\rho\beta\eta}{\to} M_1$, $Px^L \stackrel{\rho\beta\eta}{\to} P'$ and $\forall L \in \mathcal{L}_{\mathbb{N}}, x^L \notin \mathrm{fv}(P)$ and $L \succeq \mathrm{d}(P)$. By lemma 23 there are two cases:
  - ∗ $P' = P''x^L$ and $P \stackrel{\rho\beta\eta}{\to} P''$. By IH, there is M' such that $P'' \stackrel{\rho\beta\eta}{\to} M' \stackrel{\rho\beta\eta}{\leftarrow} M_1$. By lemma 21.2, $\forall L \in \mathcal{L}_{\mathbb{N}}, x^L \notin \mathrm{fv}(P'')$ and $L \succeq \mathrm{d}(P'')$, hence, $\lambda x^L.P' = \lambda x^L.P''x^L \stackrel{\rho\beta\eta}{\to} M' \stackrel{\rho\beta\eta}{\leftarrow} M_1$.
  - ∗ $P = \lambda y^L.Q$, $Q \stackrel{\rho\beta\eta}{\to} Q'$, $Q \diamond x^L$ and $P' = Q'[y^L := x^L]$. So we have $M_1 \stackrel{\rho\beta\eta}{\leftarrow} \lambda x^L.(\lambda y^L.Q)x^L \stackrel{\rho\beta\eta}{\to} \lambda x^L.Q'[y^L := x^L]$ where $M_1 \stackrel{\rho\beta\eta}{\leftarrow} \lambda y^L.Q = \lambda x^L.Q[y^L := x^L]$ since $\forall L \in \mathcal{L}_{\mathbb{N}}, x^L \notin \mathrm{fv}(P)$.

    By lemma 22.2, $\lambda x^L.Q[y^L := x^L] \stackrel{\rho\beta\eta}{\to} \lambda x^L.Q'[y^L := x^L]$. Hence by IH, there is M' such that $M_1 \stackrel{\rho\beta\eta}{\to} M' \stackrel{\rho\beta\eta}{\leftarrow} \lambda x^L.Q'[y^L := x^L]$.

– Let $r = \beta$:

- If $M = x^L$, by lemma 23, $M_1 = M_2 = x^L$. Take $M' = x^L$.
- If $N_2P_2 \stackrel{\rho\beta}{\leftarrow} NP \stackrel{\rho\beta}{\to} N_1P_1$ where $N_2 \stackrel{\rho\beta}{\leftarrow} N \stackrel{\rho\beta}{\to} N_1$, $P_2 \stackrel{\rho\beta}{\leftarrow} P \stackrel{\rho\beta}{\to} P_1$ and $N \diamond P$, then, by IH, $\exists N', P'$ such that $N_2 \stackrel{\rho\beta}{\to} N' \stackrel{\rho\beta}{\leftarrow} N_1$ and $P_2 \stackrel{\rho\beta}{\to} P' \stackrel{\rho\beta}{\leftarrow} P_1$. By lemma 21.3, $N_1 \diamond P_1$ and $N_2 \diamond P_2$. Hence, $N_2P_2 \stackrel{\rho\beta}{\to} N'P' \stackrel{\rho\beta}{\leftarrow} N_1P_1$.
- If $(\lambda x^L.P_1)Q_1 \stackrel{\rho\beta}{\leftarrow} (\lambda x^L.P)Q \stackrel{\rho\beta}{\to} P_2[x^L := Q_2]$ where $\lambda x^L.P \stackrel{\rho\beta}{\to} \lambda x^L.P_1$, $P \stackrel{\rho\beta}{\to} P_2$, $Q_1 \stackrel{\rho\beta}{\leftarrow} Q \stackrel{\rho\beta}{\to} Q_2$, $\mathrm{d}(Q) = L$, $P \diamond Q$ and $(\lambda x^L.P) \diamond Q$, then, by lemma 23, $P \stackrel{\rho\beta}{\to} P_1$. By IH, $\exists P', Q'$ such that $P_1 \stackrel{\rho\beta}{\to} P' \stackrel{\rho\beta}{\leftarrow} P_2$ and $Q_1 \stackrel{\rho\beta}{\to} Q' \stackrel{\rho\beta}{\leftarrow} Q_2$. By lemma 21.2, $\mathrm{d}(Q_1) = \mathrm{d}(Q_2) = \mathrm{d}(Q) = L$. By lemma 21.3, $P_1 \diamond Q_1$. Hence, $(\lambda x^L.P_1)Q_1 \stackrel{\rho\beta}{\to} P'[x^L := Q']$.

  Moreover, since $P_2 \stackrel{\rho\beta}{\to} P'$, $Q_2 \stackrel{\rho\beta}{\to} Q'$, $\mathrm{d}(Q_2) = L$ and by lemma 21.3, $P_2 \diamond Q_2$., then, by lemma 22.2, $P_2[x^L := Q_2] \stackrel{\rho\beta}{\to} P'[x^L := Q']$.
- If $P_1[x^L := Q_1] \stackrel{\rho\beta}{\leftarrow} (\lambda x^L.P)Q \stackrel{\rho\beta}{\to} P_2[x^L := Q_2]$ where $P_1 \stackrel{\rho\beta}{\leftarrow} P \stackrel{\rho\beta}{\to} P_2$, $Q_1 \stackrel{\rho\beta}{\leftarrow} Q \stackrel{\rho\beta}{\to} Q_2$, $\mathrm{d}(Q) = L$ and $P \diamond Q$ then by IH, $\exists P', Q'$ where $P_1 \stackrel{\rho\beta}{\to} P' \stackrel{\rho\beta}{\leftarrow} P_2$ and $Q_1 \stackrel{\rho\beta}{\to} Q' \stackrel{\rho\beta}{\leftarrow} Q_2$. By lemma 21.2, $\mathrm{d}(Q_1) = \mathrm{d}(Q_2) = \mathrm{d}(Q) = L$. By lemma 21.3, $P_1 \diamond Q_1$ and $P_2 \diamond Q_2$. Hence, by lemma 22.2, $P_1[x^L := Q_1] \stackrel{\rho\beta}{\to} P'[x^L := Q'] \stackrel{\rho\beta}{\leftarrow} P_2[x^L := Q_2]$.
- If $\lambda x^L.N_2 \stackrel{\rho\beta}{\leftarrow} \lambda x^L.N \stackrel{\rho\beta}{\to} \lambda x^L.N_1$ where $N_2 \stackrel{\rho\beta}{\leftarrow} N \stackrel{\rho\beta}{\to} N_1$, by IH, there is N' such that $N_2 \stackrel{\rho\beta}{\to} N' \stackrel{\rho\beta}{\leftarrow} N_1$. Hence, $\lambda x^L.N_2 \stackrel{\rho\beta}{\to} \lambda x^L.N' \stackrel{\rho\beta}{\leftarrow} \lambda x^L.N_1$.

2. First show by induction on $M \stackrel{\rho_r}{\twoheadrightarrow} M_1$ (and using 1) that if $M_2 \stackrel{\rho_r}{\twoheadleftarrow} M \stackrel{\rho_r}{\twoheadrightarrow} M_1$, then there is M' such that $M_2 \stackrel{\rho_r}{\twoheadrightarrow} M' \stackrel{\rho_r}{\twoheadleftarrow} M_1$. Then use this to show 2 by induction on $M \stackrel{\rho_r}{\twoheadrightarrow} M_2$. □

Proof (Of Theorem 2).

1. For $r \in \{\beta, \beta\eta\}$, by lemma 24.2, $\stackrel{\rho_r}{\twoheadrightarrow}$ is confluent. by lemma 21.1 and 21.2, $M \stackrel{\rho_r}{\twoheadrightarrow} N$ iff $M \rhd_r^* N$. Then $\rhd_r^*$ is confluent.
   For $r = h$, since if $M \rhd_r^* M_1$ and $M \rhd_r^* M_2$, $M_1 = M_2$, we take $M' = M_1$.
2. If) is by definition of $\simeq_r$. Only if) is by induction on $M_1 \simeq_r M_2$ using 1. □

B Proofs of section 3

Proof (Of lemma 2).

1. By definition.
2. By induction on U.
   – If $U = a$ ($\mathrm{d}(U) = \oslash$), nothing to prove.
   – If $U = V \to T$ ($\mathrm{d}(U) = \oslash$), nothing to prove.
   – If $U = \omega^L$, nothing to prove.
   – If $U = U_1 \sqcap U_2$ ($\mathrm{d}(U) = \mathrm{d}(U_1) = \mathrm{d}(U_2) = L$), by IH we have four cases:
     • If $U_1 = U_2 = \omega^L$ then $U = \omega^L$.
     • If $U_1 = \omega^L$ and $U_2 = \boldsymbol{e}_L \sqcap_{i=1}^k T_i$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $T_i \in \mathbb{T}$ then $U = U_2$ (since ω^L is a neutral).
     • If $U_2 = \omega^L$ and $U_1 = \boldsymbol{e}_L \sqcap_{i=1}^k T_i$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $T_i \in \mathbb{T}$ then $U = U_1$ (since ω^L is a neutral).
     • If $U_1 = \boldsymbol{e}_L \sqcap_{i=1}^p T_i$ and $U_2 = \boldsymbol{e}_L \sqcap_{i=p+1}^{p+q} T_i$ where $p, q \geq 1$, $\forall 1 \leq i \leq p+q$, $T_i \in \mathbb{T}$ then $U = \boldsymbol{e}_L \sqcap_{i=1}^{p+q} T_i$.
   – If $U = \overline{e}_{n_1} V$ ($L = \mathrm{d}(U) = n_1 :: \mathrm{d}(V) = n_1 :: K$), by IH we have two cases:
     • If $V = \omega^K$, $U = \overline{e}_{n_1}\omega^K = \omega^L$.
     • If $V = \boldsymbol{e}_K \sqcap_{i=1}^p T_i$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $T_i \in \mathbb{T}$ then $U = \boldsymbol{e}_L \sqcap_{i=1}^p T_i$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $T_i \in \mathbb{T}$.
3. (a) By induction on $U_1 \sqsubseteq U_2$.
   (b) By induction on $U_1 \sqsubseteq U_2$.
   (c) By induction on K. We do the induction step. Let $U_1 = \overline{e}_i U$. By induction on $\overline{e}_i U \sqsubseteq U_2$ we obtain $U_2 = \overline{e}_i U'$ and $U \sqsubseteq U'$.
   (d) same proof as in the previous item.
   (e) By induction on $U_1 \sqsubseteq U_2$:
     – By ref, $U_1 = U_2$.
     – If $\dfrac{\sqcap_{i=1}^p \boldsymbol{e}_K(U_i \to T_i) \sqsubseteq U \qquad U \sqsubseteq U_2}{\sqcap_{i=1}^p \boldsymbol{e}_K(U_i \to T_i) \sqsubseteq U_2}$. If $U = \omega^K$ then by (b), $U_2 = \omega^K$. If $U = \sqcap_{j=1}^q \boldsymbol{e}_K(U'_j \to T'_j)$ where $q \geq 1$ and $\forall 1 \leq j \leq q$, $\exists 1 \leq i \leq p$ such that $U'_j \sqsubseteq U_i$ and $T_i \sqsubseteq T'_j$ then by IH, $U_2 = \omega^K$ or $U_2 = \sqcap_{k=1}^r \boldsymbol{e}_K(U''_k \to T''_k)$ where $r \geq 1$ and $\forall 1 \leq k \leq r$, $\exists 1 \leq j \leq q$ such that $U''_k \sqsubseteq U'_j$ and $T'_j \sqsubseteq T''_k$. Hence, by tr, $\forall 1 \leq k \leq r$, $\exists 1 \leq i \leq p$ such that $U''_k \sqsubseteq U_i$ and $T_i \sqsubseteq T''_k$.

- By $\sqcap_E$, $U_2 = \omega^K$ or $U_2 = \sqcap_{j=1}^{q} \boldsymbol{e}_K(U'_j \to T'_j)$ where $1 \le q \le p$ and $\forall 1 \le j \le q$, $\exists 1 \le i \le p$ such that $U_i = U'_j$ and $T_i = T'_j$.
- Case $\sqcap$ is by IH.
- Case $\to$ is trivial.
- If $\dfrac{\sqcap_{i=1}^{p} \boldsymbol{e}_L(U_i \to T_i) \sqsubseteq U_2}{\sqcap_{i=1}^{p} \boldsymbol{e}_K(U_i \to T_i) \sqsubseteq \overline{e}_i U_2}$ where $K = i :: L$ then by IH, $U_2 = \omega^L$ and so $\overline{e}_i U_2 = \omega^K$ or $U_2 = \sqcap_{j=1}^{q} \boldsymbol{e}_L(U'_j \to T'_j)$ so $\overline{e}_i U_2 = \sqcap_{j=1}^{q} \boldsymbol{e}_K(U'_j \to T'_j)$ where $q \ge 1$ and $\forall 1 \le j \le q$, $\exists 1 \le i \le p$ such that $U'_j \sqsubseteq U_i$ and $T_i \sqsubseteq T'_j$.

4. By $\sqcap_E$ and since ω^L is a neutral.
5. By induction on $U \sqsubseteq U'_1 \sqcap U'_2$.
   - Let $\dfrac{}{U'_1 \sqcap U'_2 \sqsubseteq U'_1 \sqcap U'_2}$. By ref, $U'_1 \sqsubseteq U'_1$ and $U'_2 \sqsubseteq U'_2$.
   - Let $\dfrac{U \sqsubseteq U'' \qquad U'' \sqsubseteq U'_1 \sqcap U'_2}{U \sqsubseteq U'_1 \sqcap U'_2}$. By IH, $U'' = U''_1 \sqcap U''_2$ such that $U''_1 \sqsubseteq U'_1$ and $U''_2 \sqsubseteq U'_2$. Again by IH, $U = U_1 \sqcap U_2$ such that $U_1 \sqsubseteq U''_1$ and $U_2 \sqsubseteq U''_2$. So by tr, $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$.
   - Let $\dfrac{}{(U'_1 \sqcap U'_2) \sqcap U \sqsubseteq U'_1 \sqcap U'_2}$. By ref, $U'_1 \sqsubseteq U'_1$ and $U'_2 \sqsubseteq U'_2$. Moreover $\mathrm{d}(U) = \mathrm{d}(U'_1 \sqcap U'_2) = \mathrm{d}(U'_1)$ then by $\sqcap_E$, $U'_1 \sqcap U \sqsubseteq U'_1$.
   - If $\dfrac{U_1 \sqsubseteq U'_1 \quad \& \quad U_2 \sqsubseteq U'_2}{U_1 \sqcap U_2 \sqsubseteq U'_1 \sqcap U'_2}$ there is nothing to prove.
   - $\dfrac{V_2 \sqsubseteq V_1 \quad \& \quad T_1 \sqsubseteq T_2}{V_1 \to T_1 \sqsubseteq V_2 \to T_2}$ then $U'_1 = U'_2 = V_2 \to T_2$ and $U = U_1 \sqcap U_2$ such that $U_1 = U_2 = V_1 \to T_1$ and we are done.
   - If $\dfrac{U \sqsubseteq U'_1 \sqcap U'_2}{eU \sqsubseteq eU'_1 \sqcap eU'_2}$ then by IH $U = U_1 \sqcap U_2$ such that $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$. So, $eU = eU_1 \sqcap eU_2$ and by $\sqsubseteq_e$, $eU_1 \sqsubseteq eU'_1$ and $eU_2 \sqsubseteq eU'_2$.
6. By induction on $\Gamma \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2$.
   - Let $\dfrac{}{\Gamma'_1 \sqcap \Gamma'_2 \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2}$. By ref, $\Gamma'_1 \sqsubseteq \Gamma'_1$ and $\Gamma'_2 \sqsubseteq \Gamma'_2$.
   - Let $\dfrac{\Gamma \sqsubseteq \Gamma'' \qquad \Gamma'' \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2}{\Gamma \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2}$. By IH, $\Gamma'' = \Gamma''_1 \sqcap \Gamma''_2$ such that $\Gamma''_1 \sqsubseteq \Gamma'_1$ and $\Gamma''_2 \sqsubseteq \Gamma'_2$. Again by IH, $\Gamma = \Gamma_1 \sqcap \Gamma_2$ such that $\Gamma_1 \sqsubseteq \Gamma''_1$ and $\Gamma_2 \sqsubseteq \Gamma''_2$. So by tr, $\Gamma_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$.
   - Let $\dfrac{U_1 \sqsubseteq U_2}{\Gamma, (y^n : U_1) \sqsubseteq \Gamma, (y^n : U_2)}$ where $\Gamma, (y^n : U_2) = \Gamma'_1 \sqcap \Gamma'_2$.
     - If $\Gamma'_1 = \Gamma''_1, (y^n : U'_2)$ and $\Gamma'_2 = \Gamma''_2, (y^n : U''_2)$ such that $U_2 = U'_2 \sqcap U''_2$, then by 5, $U_1 = U'_1 \sqcap U''_1$ such that $U'_1 \sqsubseteq U'_2$ and $U''_1 \sqsubseteq U''_2$. Hence $\Gamma = \Gamma''_1 \sqcap \Gamma''_2$ and $\Gamma, (y^n : U_1) = \Gamma_1 \sqcap \Gamma_2$ where $\Gamma_1 = \Gamma''_1, (y^n : U'_1)$ and $\Gamma_2 = \Gamma''_2, (y^n : U''_1)$ such that $\Gamma_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$ by $\sqsubseteq_c$.
     - If $y^n \notin \mathrm{dom}(\Gamma'_1)$ then $\Gamma = \Gamma'_1 \sqcap \Gamma''_2$ where $\Gamma''_2, (y^n : U_2) = \Gamma'_2$. Hence, $\Gamma, (y^n : U_1) = \Gamma'_1 \sqcap \Gamma_2$ where $\Gamma_2 = \Gamma''_2, (y^n : U_1)$. By ref and $\sqsubseteq_c$, $\Gamma'_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$.
     - If $y^n \notin \mathrm{dom}(\Gamma'_2)$ then similar to the above case. □

Proof (Of lemma 3). 1. By definition, if $\mathrm{fv}(M) = \{x_1^{L_1}, \ldots, x_n^{L_n}\}$ then $env_M^{\omega} = (x_i^{L_i} : \omega^{L_i})_n$ and by definition, for all $i \in \{1, \ldots, n\}$, $\mathrm{d}(\omega^{L_i}) = L_i$. Moreover, if

$x^L : U, x^L : V \in env^{\omega}_M$, then $U = \omega^L = V$.
2. First show by induction on the derivation $\Gamma \sqsubseteq \Gamma'$ that if $\Gamma \sqsubseteq \Gamma'$ and $\Gamma, (x^L : U)$ is an environment, then $\Gamma, (x^L : U) \sqsubseteq \Gamma', (x^L : U)$. Then use (tr) and $(\sqsubseteq_c)$.
3. Only if) By induction on the derivation $\Gamma \sqsubseteq \Gamma'$. If) By induction on n using 2.
4. Only if) By induction on the derivation $\langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle$. If) By $\sqsubseteq_{\langle\rangle}$.
5. Let $\mathrm{fv}(M) = \{x_1^{L_1}, \dots, x_n^{L_n}\}$ and $\Gamma = (x_i^{L_i} : U_i)_n$. By definition, $env^{\omega}_M = (x_i^{L_i} : \omega^{L_i})_n$. Because $\mathrm{OK}(\Gamma)$, then for all $i \in \{1, \dots, n\}$, $\mathrm{d}(U_i) = L_i$. Hence, by lemma 2.4 and 3, $\Gamma \sqsubseteq env^{\omega}_M$.
6. Let $x^{L_1} \in \mathrm{dom}(\Gamma^{-K})$ and $x^{L_2} \in \mathrm{dom}(\Delta^{-K})$, then $x^{K::L_1} \in \mathrm{dom}(\Gamma)$ and $x^{K::L_2} \in \mathrm{dom}(\Delta)$, hence $K :: L_1 = K :: L_2$ and so $L_1 = L_2$.
7. Let $\mathrm{d}(U) = L = K :: K'$. By lemma 2:

- If $U = \omega^L$ then by lemma 2.3b, $U' = \omega^L$ and by ref, $U^{-K} = \omega^{K'} \sqsubseteq \omega^{K'} = U'^{-K}$.
- If $U = \boldsymbol{e}_L \sqcap_{i=1}^p T_i$ where $p \geq 1$ and $\forall\ 1 \leq i \leq p$, $T_i \in \mathbb{T}$ then by lemma 2.3c, $U' = \boldsymbol{e}_L V$ and $\sqcap_{i=1}^p T_i \sqsubseteq V$. Hence, by $\sqsubseteq_e$, $U^{-K} = \boldsymbol{e}_{K'} \sqcap_{i=1}^p T_i \sqsubseteq \boldsymbol{e}_{K'} V = U'^{-K}$.

8. Let $\Gamma = (x_i^{L_i} : U_i)_n$, so by lemma 3.3, $\Gamma' = (x_i^{L_i} : U'_i)_n$ and $\forall 1 \leq i \leq n$, $U_i \sqsubseteq U'_i$. Because $\mathrm{d}(\Gamma) \succeq K$, then by definition $\forall 1 \leq i \leq n$, $\mathrm{d}(U_i) \succeq K$. By lemma 3.7, $\forall i \in \{1, \dots, n\}$, $U_i^{-K} \sqsubseteq {U'}_i^{-K}$ and by lemma 3.3, $\Gamma^{-K} \sqsubseteq \Gamma'^{-K}$.
9. Let $\Gamma_1 = (x_i^{L_i} : U_i)_n, \Gamma'_1$ and $\Gamma_2 = (x_i^{L_i} : U'_i)_n, \Gamma'_2$ such that $\mathrm{dom}(\Gamma'_1) \cap \mathrm{dom}(\Gamma'_2)$. Then, by hypotheses, for all $i \in \{1, \dots, n\}$, $\mathrm{d}(U_i) = L_i = \mathrm{d}(U'_i)$. Then $\Gamma_1 \sqcap \Gamma_2 = (x_i^{L_i} : U_i \sqcap U'_i)_n, \Gamma'_1, \Gamma'_2$ is well defined. Moreover, for all $x^L : U \in \Gamma'_1$, $\mathrm{d}(U) = L$ and for all $x^L : U \in \Gamma'_2$, $\mathrm{d}(U) = L$ and for all $i \in \{1, \dots, n\}$, $\mathrm{d}(U_i \sqcap U'_i) = \mathrm{d}(U_i) = L_i = \mathrm{d}(U'_i)$.
10. Let $\Gamma = (x_j^{L_j} : U_j)_n$ then by hypothesis, for all $j \in \{1, \dots, n\}$, $\mathrm{d}(U_j) = L_j$ and $\overline{e}_i \Gamma = (x_j^{i::L_j} : \overline{e}_i U_j)$. So, for all $j \in \{1, \dots, n\}$, $\mathrm{d}(\overline{e}_i U_j) = i :: \mathrm{d}(U_j) = i :: L_j$.
11. By lemma 3.3, $\Gamma_1 = (x_i^{L_i} : U_i)_n$ and $\Gamma_2 = (x_i^{L_i} : U'_i)_n$ and for all $i \in \{1, \dots, n\}$, $U_i \sqsubseteq U'_i$. By lemma 2.3a, for all $i \in \{1, \dots, n\}$, $\mathrm{d}(U_i) = \mathrm{d}(U'_i)$. Assume $\mathrm{d}(\Gamma_1) \succeq K$ then for all $i \in \{1, \dots, n\}$, $\mathrm{d}(U_i) = \mathrm{d}(U'_i) \succeq K$ and $L_i \succeq K$, so $\mathrm{d}(\Gamma_2) \succeq K$. Assume $\mathrm{d}(\Gamma_2) \succeq K$ then for all $i \in \{1, \dots, n\}$, $\mathrm{d}(U_i) = \mathrm{d}(U'_i) \succeq K$ and $L_i \succeq K$, so $\mathrm{d}(\Gamma_1) \succeq K$. Assume $\mathrm{OK}(\Gamma_1)$ then for all $i \in \{1, \dots, n\}$, $L_i = \mathrm{d}(U_i) = \mathrm{d}(U'_i)$, so $\mathrm{OK}(\Gamma_2)$. Assume $\mathrm{OK}(\Gamma_2)$ then for all $i \in \{1, \dots, n\}$, $L_i = \mathrm{d}(U'_i) = \mathrm{d}(U_i)$, so $\mathrm{OK}(\Gamma_1)$. □

Proof (Of theorem 3).

1. and 2. By lemma 4.2 and lemma 3.3, $\Gamma \diamond \Gamma$.
   - If $\overline{x^{\oslash} : \langle (x^{\oslash} : T) \vdash T \rangle}$ then, by hypothesis, $T \in \mathbb{T} \subseteq \mathbb{U}$ and $\mathrm{d}(T) = \oslash = \mathrm{d}(x^{\oslash})$. So, $\mathrm{OK}((x^{\oslash} : T))$ and $x^{\oslash} \in \mathcal{M}$.
   - If $\overline{M : \langle env^{\omega}_M \vdash \omega^{\mathrm{d}(M)} \rangle}$. By definition, M is defined to range over $\mathcal{M}$ and $\mathrm{OK}(env^{\omega}_M)$ by lemma 3.1. By definition, $\omega^{\mathrm{d}(M)} \in \mathbb{U}$. Let $\mathrm{fv}(M) = \{x^{L_1}, \dots, x^{L_n}\}$, so $env^{\omega}_M = (x_i^{L_i} : \omega^{L_i})_n$ and by lemma 18.4, $\forall 1 \leq i \leq n, L_i \succeq \mathrm{d}(M) = \mathrm{d}(\omega^{\mathrm{d}(M)})$.

– If $\dfrac{M : \langle \Gamma, (x^L : U) \vdash T\rangle}{\lambda x^L.M : \langle \Gamma \vdash U \to T\rangle}$ then by IH, $M \in \mathcal{M}$, $T \in \mathbb{U}$ $\Gamma, (x^L : U) \in Env$, $\mathrm{OK}(\Gamma, (x^L : U))$ and $\mathrm{d}(\Gamma, (x^L : U)) \succeq \mathrm{d}(T) = \mathrm{d}(M)$. By hypothesis, $T \in \mathbb{T}$. Because $\Gamma, (x^L : U) \in Env$, then $U \in \mathbb{U}$. So $U \to T \in \mathbb{T} \subset \mathbb{U}$. Let $\Gamma = (x_i^{L_i} : U_i)_n$, then for all $i \in \{1, \ldots, n\}$, $L_i = \mathrm{d}(U_i) \succeq \mathrm{d}(T) = \mathrm{d}(U \to T)$ and $\mathrm{d}(U) = L \succeq \mathrm{d}(M)$. Hence, $\lambda x^L.M \in \mathcal{M}$ and $\mathrm{OK}(\Gamma)$. So, $\mathrm{d}(\lambda x^L.M) = \mathrm{d}(M) = \mathrm{d}(T) = \mathrm{d}(U \to T)$.
– If $\dfrac{M : \langle \Gamma \vdash T\rangle \quad x^L \not\in \mathrm{dom}(\Gamma)}{\lambda x^L.M : \langle \Gamma \vdash \omega^L \to T\rangle}$ then by IH, $M \in \mathcal{M}$, $T \in \mathbb{U}$, $\Gamma \in Env$, $\mathrm{OK}(\Gamma)$ and $\mathrm{d}(\Gamma) \succeq \mathrm{d}(T) = \mathrm{d}(M)$. By hypothesis, $T \in \mathbb{T}$. So $\mathrm{d}(T) = \oslash = \mathrm{d}(M) \preceq L$. By definition, $\omega^L \in \mathbb{U}$. So, $\omega^L \to T \in \mathbb{T} \subset \mathbb{U}$. So, $\lambda x^L.M \in \mathcal{M}$ and $\mathrm{d}(\lambda x^L.M) = \mathrm{d}(M) = \mathrm{d}(T) = \mathrm{d}(\omega^L \to T)$.
– If $\dfrac{M_1 : \langle \Gamma_1 \vdash U \to T\rangle \qquad M_2 : \langle \Gamma_2 \vdash U\rangle \qquad \Gamma_1 \diamond \Gamma_2}{M_1 M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle}$ then by IH, $M_1, M_2 \in \mathcal{M}$, $\Gamma_1, \Gamma_1 \in Env$, $U \to T, U \in \mathbb{U}$, $\mathrm{OK}(\Gamma_1)$, $\mathrm{OK}(\Gamma_2)$ and $\mathrm{d}(\Gamma_1) \succeq \mathrm{d}(U \to T) = \mathrm{d}(M_1)$ and $\mathrm{d}(\Gamma_2) \succeq \mathrm{d}(U) = \mathrm{d}(M_2)$. By definition, $\Gamma_1 \sqcap \Gamma_2$ is a type environment. By hypothesis, $T \in \mathbb{T} \subset \mathbb{U}$. By lemma 3.9 and lemma 4.3, $\mathrm{OK}(\Gamma_1 \sqcap \Gamma_2)$ and $M_1 \diamond M_2$. Because $\mathrm{d}(M_2) = \mathrm{d}(U) \succeq \oslash = \mathrm{d}(U \to T) = \mathrm{d}(M_1)$, then $M_1 M_2 \in \mathcal{M}$. We have trivially, $\mathrm{d}(\Gamma_1 \sqcap \Gamma_1) \succeq \oslash$. Moreover $\mathrm{d}(M_1 M_2) = \mathrm{d}(M_1) = \mathrm{d}(U \to T) = \mathrm{d}(T)$.
– If $\dfrac{M : \langle \Gamma \vdash U_1\rangle \qquad M : \langle \Gamma \vdash U_2\rangle}{M : \langle \Gamma \vdash U_1 \sqcap U_2\rangle}$ then by IH, $M \in \mathcal{M}$, $\Gamma \in Env$, $U_1, U_2 \in \mathbb{U}$, $\mathrm{OK}(\Gamma)$ and $\mathrm{d}(\Gamma) \succeq \mathrm{d}(U_1) = \mathrm{d}(M)$ and $\mathrm{d}(\Gamma) \succeq \mathrm{d}(U_2) = \mathrm{d}(M)$. So $\mathrm{d}(U_1) = \mathrm{d}(M) = \mathrm{d}(U_2)$. Hence, $U_1 \sqcap U_2 \in \mathbb{U}$. Moreover, $\mathrm{d}(\Gamma) \succeq \mathrm{d}(U_1) = \mathrm{d}(U_1 \sqcap U_2) = \mathrm{d}(M)$.
– If $\dfrac{M : \langle \Gamma \vdash U\rangle}{M^{+k} : \langle \overline{e}_k \Gamma \vdash \overline{e}_k U\rangle}$ then by IH, $M \in \mathcal{M}$, $\Gamma \in Env$, $U \in \mathbb{U}$, $\mathrm{OK}(\Gamma)$ and $\mathrm{d}(\Gamma) \succeq \mathrm{d}(U) = \mathrm{d}(M)$. Then, by definition, $\overline{e}_k U \in \mathbb{U}$. By definition, $\overline{e}_k \Gamma \in Env$. Then, by lemma 19.1 and lemma 3.10, $M^{+i} \in \mathcal{M}$ and $\mathrm{OK}(\overline{e}_k \Gamma)$. Let $\Gamma = (x_j^{L_j} : U_j)_n$ so $\overline{e}_k \Gamma = (x_j^{k::L_j} : \overline{e}_k U_j)_n$ and for all $j \in \{1, \ldots, n\}$, because $\mathrm{d}(U_j) = L_j \succeq \mathrm{d}(U)$ then $\mathrm{d}(\overline{e}_k U_j) = k :: \mathrm{d}(U_j) = k :: L_j \succeq k :: \mathrm{d}(U) = \mathrm{d}(\overline{e}_k U) = k :: \mathrm{d}(M) =^{19.1} \mathrm{d}(M^{+k})$.
– If $\dfrac{M : \langle \Gamma \vdash U\rangle \qquad \langle \Gamma \vdash U\rangle \sqsubseteq \langle \Gamma' \vdash U'\rangle}{M : \langle \Gamma' \vdash U'\rangle}$ then by IH, $M \in \mathcal{M}$, $U \in \mathbb{U}$, $\Gamma \in Env$, $\mathrm{OK}(\Gamma)$ and $\mathrm{d}(\Gamma) \succeq \mathrm{d}(U) = \mathrm{d}(M)$. By lemma 3.4, $\Gamma' \sqsubseteq \Gamma$, hence, $\Gamma' \in Env$. By lemma 3.11, $\mathrm{OK}(\Gamma')$. Let $\Gamma = (x_i^{L_i} : U_i)_n$, so $\forall 1 \leq i \leq n, \mathrm{d}(U_i) = L_i \succeq \mathrm{d}(U)$. By lemma 3.3, $\Gamma' = (x_i^{L_i} : U'_i)_n$ and $\forall 1 \leq i \leq n$, $U_i \sqsubseteq U'_i$ so by lemma 2.3a, $\mathrm{d}(U_i) = \mathrm{d}(U'_i)$. By lemma 3.4, $U \sqsubseteq U'$ so by lemma 2.3a, $\mathrm{d}(U) = \mathrm{d}(U')$. Hence $\forall 1 \leq i \leq n, \mathrm{d}(U'_i) = L_i \succeq \mathrm{d}(U') = \mathrm{d}(M)$.

3. By induction on $M : \langle \Gamma \vdash U\rangle$. Case $K = \oslash$ is trivial, consider $K = i :: K'$. Let $\mathrm{d}(U) = K :: L$. Since $\mathrm{d}(U) \succeq K$, U^{-K} is well defined. Since by 1. $\mathrm{d}(\Gamma) \succeq \mathrm{d}(U) = \mathrm{d}(M)$, M^{-K} and Γ^{-K} are well defined too.
   – If $\dfrac{}{M : \langle env_M^\omega \vdash \omega^{\mathrm{d}(M)}\rangle}$. By ω, $M^{-K} : \langle env_{M^{-K}}^\omega \vdash \omega^L\rangle$.
   – $\sqcap_I$ is by IH.

– If $\dfrac{M : \langle \Gamma \vdash U\rangle}{M^{+i} : \langle \overline{e}_i\Gamma \vdash \overline{e}_i U\rangle}$. Since $\mathrm{d}(\overline{e}_i U) = i :: K' :: L$, $\mathrm{d}(U) = K' :: L$, so $\mathrm{d}(U) \succeq K'$ and by IH, $M^{-K'} : \langle \Gamma^{-K'} \vdash U^{-K'}\rangle$, so by e and lemma 19.4, $(M^{+i})^{-K} : \langle (\overline{e}_i\Gamma)^{-K} \vdash (\overline{e}_i U)^{-K}\rangle$.
– If $\dfrac{M : \langle \Gamma \vdash U\rangle \qquad \langle \Gamma \vdash U\rangle \sqsubseteq \langle \Gamma' \vdash U'\rangle}{M : \langle \Gamma' \vdash U'\rangle}$ then by lemma 3.4, $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$. By lemma 2.3a, $\mathrm{d}(U) = \mathrm{d}(U') \succeq K$. By IH, $M^{-K} : \langle \Gamma^{-K} \vdash U^{-K}\rangle$. Hence by lemma 3.11, lemma 3.7, lemma 3.8 and $\sqsubseteq$, $M^{-K} : \langle \Gamma'^{-K} \vdash U'^{-K}\rangle$. □

Proof (Of remark 1).

1. Let $M : \langle \Gamma_1 \vdash U_1\rangle$ and $M : \langle \Gamma_2 \vdash U_2\rangle$. By lemma 4.2, $\mathrm{dom}(\Gamma_1) = \mathrm{fv}(M) = \mathrm{dom}(\Gamma_2)$. Let $\Gamma_1 = (x_i^{L_i} : V_i)_n$ and $\Gamma_2 = (x_i^{L_i} : V_i')_n$. Then, by lemma 3.2, $\forall 1 \leq i \leq n$, $\mathrm{d}(V_i) = \mathrm{d}(V_i') = L_i$. By $\sqcap_E$, $V_i \sqcap V_i' \sqsubseteq V_i$ and $V_i \sqcap V_i' \sqsubseteq V_i'$. Hence, by lemma 3.3, $\Gamma_1 \sqcap \Gamma_2 \sqsubseteq \Gamma_1$ and $\Gamma_1 \sqcap \Gamma_2 \sqsubseteq \Gamma_2$ and by $\sqsubseteq$ and $\sqsubseteq_{\langle\rangle}$, $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U_1\rangle$ and $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U_2\rangle$. Finally, by $\sqcap_I$, $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U_1 \sqcap U_2\rangle$.
2. By lemma 2, either $U = \omega^L$ so by ω, $x^L : \langle (x^L : \omega^L) \vdash \omega^L\rangle$.Or $U = \sqcap_{i=1}^p \boldsymbol{e}_L T_i$ where $p \geq 1$, and $\forall 1 \leq i \leq p$, $T_i \in \mathbb{T}$. Let $1 \leq i \leq p$.By ax, $x^{\oslash} : \langle (x^{\oslash} : T_i) \vdash T_i\rangle$, hence by e, $x^L : \langle (x^L : \boldsymbol{e}_L T_i) \vdash \boldsymbol{e}_L T_i\rangle$. Now, by $\sqcap_I'$, $x^L : \langle (x^L : U) \vdash U\rangle$. □

C Proofs of section 4

Proof (Of lemma 5). 1. By induction on the derivation $x^L : \langle \Gamma \vdash U\rangle$. We have fives cases:

– If $\dfrac{}{x^{\oslash} : \langle (x^{\oslash} : T) \vdash T\rangle}$ then it is done using (ref).
– If $\dfrac{}{x^L : \langle (x^L : \omega^L) \vdash \omega^L\rangle}$ then it is done using (ref).
– If $\dfrac{x^L : \langle \Gamma \vdash U_1\rangle \qquad x^L : \langle \Gamma \vdash U_2\rangle}{x^L : \langle \Gamma \vdash U_1 \sqcap U_2\rangle}$. By IH, $\Gamma = (x^L : V)$, $V \sqsubseteq U_1$ and $V \sqsubseteq U_2$, then by rule $\sqcap$, $V \sqsubseteq U_1 \sqcap U_2$.
– If $\dfrac{x^L : \langle \Gamma \vdash U\rangle}{x^{i::L} : \langle \overline{e}_i\Gamma \vdash \overline{e}_i U\rangle}$. Then by IH, $\Gamma = (x^L : V)$ and $V \sqsubseteq U$, so $\overline{e}_i\Gamma = (x^{i::L} : \overline{e}_i V)$ and by $\sqsubseteq_e$, $\overline{e}_i V \sqsubseteq \overline{e}_i U$,
– If $\dfrac{x^L : \langle \Gamma' \vdash U'\rangle \quad \langle \Gamma' \vdash U'\rangle \sqsubseteq \langle \Gamma \vdash U\rangle}{x^L : \langle \Gamma \vdash U\rangle}$. By lemma 3.4, $\Gamma \sqsubseteq \Gamma'$ and $U' \sqsubseteq U$ and, by IH, $\Gamma' = (x^L : V')$ and $V' \sqsubseteq U'$. Then, by lemma 3.3, $\Gamma = (x^L : V)$, $V \sqsubseteq V'$ and, by rule tr, $V \sqsubseteq U$.

2. By induction on the derivation $\lambda x^L.M : \langle \Gamma \vdash U\rangle$. We have five cases:

– If $\dfrac{}{\lambda x^L.M : \langle env^{\omega}_{\lambda x^L.M} \vdash \omega^{\mathrm{d}(\lambda x^L.M)}\rangle}$ then it is done.

– If $\dfrac{M : \langle \Gamma, x^L : U \vdash T\rangle}{\lambda x^L.M : \langle \Gamma \vdash U \to T\rangle}$ ($\mathrm{d}(U \to T) = \oslash$) then it is done.

– If $\dfrac{\lambda x^L.M : \langle \Gamma \vdash U_1\rangle \quad \lambda x^L.M : \langle \Gamma \vdash U_2\rangle}{\lambda x^L.M : \langle \Gamma \vdash U_1 \sqcap U_2\rangle}$ then $\mathrm{d}(U_1 \sqcap U_2) = \mathrm{d}(U_1) = \mathrm{d}(U_2) = K$. By IH, we have four cases:
  - If $U_1 = U_2 = \omega^K$, then $U_1 \sqcap U_2 = \omega^K$.
  - If $U_1 = \omega^K$, $U_2 = \sqcap_{i=1}^p \boldsymbol{e}_K(V_i \to T_i)$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $M : \langle \Gamma, x^L : \boldsymbol{e}_K V_i \vdash \boldsymbol{e}_K T_i\rangle$, then $U_1 \sqcap U_2 = U_2$ (ω^K is a neutral element).
  - If $U_2 = \omega^K$, $U_1 = \sqcap_{i=1}^p \boldsymbol{e}_K(V_i \to T_i)$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $M : \langle \Gamma, x^L : \boldsymbol{e}_K V_i \vdash \boldsymbol{e}_K T_i\rangle$, then $U_1 \sqcap U_2 = U_1$ (ω^K is a neutral element).
  - If $U_1 = \sqcap_{i=1}^p \boldsymbol{e}_K(V_i \to T_i)$, $U_2 = \sqcap_{i=p+1}^{p+q} \boldsymbol{e}_K(V_i \to T_i)$ (hence $U_1 \sqcap U_2 = \sqcap_{i=1}^{p+q} \boldsymbol{e}_K(V_i \to T_i)$) where $p, q \geq 1$, $\forall 1 \leq i \leq p+q$, $M : \langle \Gamma, x^L : \boldsymbol{e}_K V_i \vdash \boldsymbol{e}_K T_i\rangle$, we are done.

– If $\dfrac{\lambda x^L.M : \langle \Gamma \vdash U\rangle}{\lambda x^{i::L}.M^{+i} : \langle \overline{e}_i \Gamma \vdash \overline{e}_i U\rangle}$. $\mathrm{d}(\overline{e}_i U) = i :: \mathrm{d}(U) = i :: K' = K$. By IH, we have two cases:
  - If $U = \omega^{K'}$ then $\overline{e}_i U = \omega^K$.
  - If $U = \sqcap_{j=1}^p \boldsymbol{e}_{K'}(V_j \to T_j)$, where $p \geq 1$ and for all $1 \leq j \leq p$, $M : \langle \Gamma, x^L : \boldsymbol{e}_{K'} V_j \vdash \boldsymbol{e}_{K'} T_j\rangle$. So $\overline{e}_i U = \sqcap_{j=1}^p \boldsymbol{e}_K(V_j \to T_j)$ and by e, for all $1 \leq j \leq p$, $M^{+i} : \langle \overline{e}_i \Gamma, x^{i::L} : \boldsymbol{e}_K V_j \vdash \boldsymbol{e}_K T_j\rangle$.

– Let $\dfrac{\lambda x^L.M : \langle \Gamma \vdash U\rangle \quad \langle \Gamma \vdash U\rangle \sqsubseteq \langle \Gamma' \vdash U'\rangle}{\lambda x^L.M : \langle \Gamma' \vdash U'\rangle}$. By lemma 3.4, $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$ and by lemma 2.3a $\mathrm{d}(U) = \mathrm{d}(U') = K$. By IH, we have two cases:
  - If $U = \omega^K$, then, by lemma 2.3b, $U' = \omega^K$.
  - If $U = \sqcap_{i=1}^p \boldsymbol{e}_K(V_i \to T_i)$, where $p \geq 1$ and for all $1 \leq i \leq p$ $M : \langle \Gamma, x^L : \boldsymbol{e}_K V_i \vdash \boldsymbol{e}_K T_i\rangle$. By lemma 2.3e:
    * Either $U' = \omega^K$.
    * Or $U' = \sqcap_{i=1}^q \boldsymbol{e}_K(V'_i \to T'_i)$, where $q \geq 1$ and $\forall 1 \leq i \leq q$, $\exists 1 \leq j_i \leq p$ such that $V'_i \sqsubseteq V_{j_i}$ and $T_{j_i} \sqsubseteq T'_i$. Let $1 \leq i \leq q$. Since, by lemma 3.4, $\langle \Gamma, x^L : \boldsymbol{e}_K V_{j_i} \vdash \boldsymbol{e}_K T_{j_i}\rangle \sqsubseteq \langle \Gamma', x^L : \boldsymbol{e}_K V'_i \vdash \boldsymbol{e}_K T'_i\rangle$, then $M : \langle \Gamma', x^L : \boldsymbol{e}_K V'_i \vdash \boldsymbol{e}_K T'_i\rangle$.

3. Similar as the proof of 2.
4. By induction on the derivation $M\, x^L : \langle \Gamma, x^L : U \vdash T\rangle$. We have two cases:

– Let $\dfrac{M : \langle \Gamma \vdash V \to T\rangle \quad x^L : \langle (x^L : U) \vdash V\rangle \quad \Gamma \diamond (x^L : U)}{M\, x^L : \langle \Gamma, (x^L : U) \vdash T\rangle}$ (where, by 1. $U \sqsubseteq V$). Since $V \to T \sqsubseteq U \to T$, we have $M : \langle \Gamma \vdash U \to T\rangle$.

– Let $\dfrac{M\, x^L : \langle \Gamma', (x^L : U') \vdash V'\rangle \quad \langle \Gamma', (x^L : U') \vdash V'\rangle \sqsubseteq \langle \Gamma, (x^L : U) \vdash V\rangle}{M\, x^L : \langle \Gamma, (x^L : U) \vdash V\rangle}$ (by lemma 3). By lemma 3, $\Gamma \sqsubseteq \Gamma'$, $U \sqsubseteq U'$ and $V' \sqsubseteq V$. By IH, $M : \langle \Gamma' \vdash U' \to V'\rangle$ and by $\sqsubseteq$, $M : \langle \Gamma \vdash U \to V\rangle$.

Proof (Of lemma 6). By lemma 3.2, $M, N \in \mathcal{M}$, $\mathrm{d}(N) = \mathrm{d}(U)$, $\mathrm{OK}(\Delta)$ and $\mathrm{OK}(\Gamma, x^L : U)$, so $\mathrm{d}(N) = \mathrm{d}(U) = L$. By lemma 3.9, $\mathrm{OK}(\Gamma \sqcap \Delta)$. By lemma 18.5, $M[x^L := N] \in \mathcal{M}$. By lemma 4.2, $x^L \in \mathrm{fv}(M)$.

We prove the lemma by induction on the derivation $M : \langle \Gamma, x^L : U \vdash V\rangle$.

– If $\dfrac{}{x^{\oslash} : \langle (x^{\oslash} : T) \vdash T\rangle}$ and $N : \langle \Delta \vdash T\rangle$, then $x^{\oslash}[x^{\oslash} := N] = N : \langle \Delta \vdash T\rangle$.
– If $\dfrac{}{M : \langle env^{\omega}_{\mathrm{fv}(M)\setminus\{x^L\}}, (x^L : \omega^L) \vdash \omega^{\mathrm{d}(M)}\rangle}$ and $N : \langle \Delta \vdash \omega^L\rangle$ then by ω, $M[x^L := N] : \langle env^{\omega}_{M[x^L:=N]} \vdash \omega^{\mathrm{d}(M[x^L:=N])}\rangle$. By lemma 18.5 $\mathrm{d}(M[x^L := N]) = \mathrm{d}(M)$. Since $x^L \in \mathrm{fv}(M)$ (and so $\mathrm{fv}(M[x^L := N]) = (\mathrm{fv}(M) \setminus \{x^L\}) \cup \mathrm{fv}(N)$), by $\sqsubseteq$, $M[x^L := N] : \langle env^{\omega}_{\mathrm{fv}(M)\setminus\{x^L\}} \sqcap \Delta \vdash \omega^{\mathrm{d}(M)}\rangle$.
– Let $\dfrac{M : \langle \Gamma, x^L : U, y^K : U' \vdash T\rangle}{\lambda y^K.M : \langle \Gamma, x^L : U \vdash U' \to T\rangle}$ where $y^K \notin \mathrm{fv}(N)\cup\{x^L\}$. So $(\lambda y^K.M)[x^L := N] = \lambda y^K.M[x^L := N]$. By lemma 18.3, $M \diamond N$. By IH, $M[x^L := N] : \langle \Gamma \sqcap \Delta, y^K : U' \vdash T\rangle$. By $\to_I$, $(\lambda y^K.M)[x^L := N] : \langle \Gamma \sqcap \Delta \vdash U' \to T\rangle$.
– Let $\dfrac{M : \langle \Gamma, x^L : U \vdash T\rangle \quad y^K \notin \mathrm{dom}(\Gamma, x^L : U)}{\lambda y^K.M : \langle \Gamma, x^L : U \vdash \omega^K \to T\rangle}$ where $y^K \notin \mathrm{fv}(N) \cup \{x^L\}$. So $(\lambda y^K.M)[x^L := N] = \lambda y^K.M[x^L := N]$. By lemma 18.3, $M \diamond N$. By lemma 4.2, $\mathrm{fv}(N) = \mathrm{dom}(\Delta)$, so $y^K \notin \mathrm{dom}(\Delta)$. By IH, $M[x^L := N] : \langle \Gamma \sqcap \Delta \vdash T\rangle$. By $\to'_I$, $(\lambda y^K.M)[x^L := N] : \langle \Gamma \sqcap \Delta \vdash \omega^K \to T\rangle$.
– Let $\dfrac{M_1 : \langle \Gamma_1, x^L : U_1 \vdash V \to T\rangle \quad M_2 : \langle \Gamma_2, x^L : U_2 \vdash V\rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1M_2 : \langle \Gamma_1 \sqcap \Gamma_2, x^L : U_1 \sqcap U_2 \vdash T\rangle}$ (by lemma 4.2) where $x^L \in \mathrm{fv}(M_1) \cap \mathrm{fv}(M_2)$, $N : \langle \Delta \vdash U_1 \sqcap U_2\rangle$. By lemma 18.3, $M_1 \diamond N$ and $M_2 \diamond N$. By $\sqcap_E$ and $\sqsubseteq$, $N : \langle \Delta \vdash U_1\rangle$ and $N : \langle \Delta \vdash U_2\rangle$. Now use IH and $\to_E$ (using the fact that $\Gamma_1 \sqcap \Delta \diamond \Gamma_2 \sqcap \Delta$, by lemma 4.2 and lemma 18.6). The cases $x^L \in \mathrm{fv}(M_1) \setminus \mathrm{fv}(M_2)$ or $x^L \in \mathrm{fv}(M_2) \setminus \mathrm{fv}(M_1)$ are similar.
– If $\dfrac{M : \langle \Gamma, x^L : U \vdash U_1\rangle \quad M : \langle \Gamma, x^L : U \vdash U_2\rangle}{M : \langle \Gamma, x^L : U \vdash U_1 \sqcap U_2\rangle}$ use IH and $\sqcap_I$.
– Let $\dfrac{M : \langle \Gamma, x^L : U \vdash V\rangle}{M^{+i} : \langle \overline{e}_i\Gamma, x^{i::L} : \overline{e}_iU \vdash \overline{e}_iV\rangle}$ and $N : \langle \Delta \vdash \overline{e}_iU\rangle$. By lemma 3.2, $\mathrm{d}(M) = \mathrm{d}(\overline{e}_iU) = i :: \mathrm{d}(U)$. By lemma 3.3, $N^{-i} : \langle \Delta^{-i} \vdash U\rangle$. By lemma 19.7 and lemma 19.2, $(N^{-i})^{+i} = N$ and $M \diamond N^{-i}$. By IH, $M[x^L := N^{-i}] : \langle \Gamma \sqcap \Delta^{-i} \vdash V\rangle$. By e and lemma 19.5, $M^{+i}[x^{i::L} := N] : \langle \overline{e}_i\Gamma \sqcap \Delta \vdash \overline{e}_iV\rangle$.
– Let $\dfrac{M : \langle \Gamma', x^L : U' \vdash V'\rangle \quad \langle \Gamma', x^L : U' \vdash V'\rangle \sqsubseteq \langle \Gamma, x^L : U \vdash V\rangle}{M : \langle \Gamma, x^L : U \vdash V\rangle}$ (lemma 3). By lemma 3, $\mathrm{dom}(\Gamma) = \mathrm{dom}(\Gamma')$, $\Gamma \sqsubseteq \Gamma'$, $U \sqsubseteq U'$ and $V' \sqsubseteq V$. Hence $N : \langle \Delta \vdash U'\rangle$ and, by IH, $M[x^L := N] : \langle \Gamma' \sqcap \Delta \vdash V'\rangle$. It is easy to show that $\Gamma \sqcap \Delta \sqsubseteq \Gamma' \sqcap \Delta$. Hence, $\langle \Gamma' \sqcap \Delta \vdash V'\rangle \sqsubseteq \langle \Gamma \sqcap \Delta \vdash V\rangle$ and $M[x^L := N] : \langle \Gamma \sqcap \Delta \vdash V\rangle$. □

The next lemma is needed in the proofs.

Lemma 25. *1. If* $\mathrm{fv}(N) \subseteq \mathrm{fv}(M)$, *then* $env^M_\omega \upharpoonright_N = env^N_\omega$.
2. *If* $\mathrm{OK}(\Gamma_1)$, $\mathrm{OK}(\Gamma_2)$, $\mathrm{fv}(M) \subseteq \mathrm{dom}(\Gamma_1)$ *and* $\mathrm{fv}(N) \subseteq \mathrm{dom}(\Gamma_2)$, *then* $(\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{MN} \sqsubseteq (\Gamma_1 \upharpoonright_M) \sqcap (\Gamma_2 \upharpoonright_N)$.
3. $\overline{e}_i(\Gamma \upharpoonright_M) = (\overline{e}_i\Gamma) \upharpoonright_{M^{+i}}$

Proof. 1. Easy. 2. First, note that $\mathrm{OK}(\Gamma_1 \sqcap \Gamma_2)$ by lemma 3.9, $\mathrm{OK}(\Gamma_1 \upharpoonright_M)$, $\mathrm{OK}(\Gamma_2 \upharpoonright_N)$ and $\mathrm{dom}((\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{MN}) = \mathrm{fv}(MN) = \mathrm{fv}(M) \cup \mathrm{fv}(N) = \mathrm{dom}(\Gamma_1 \upharpoonright_M) \cup \mathrm{dom}(\Gamma_2 \upharpoonright_N) = \mathrm{dom}((\Gamma_1 \upharpoonright_M) \sqcap (\Gamma_2 \upharpoonright_N))$. Now, we show by cases that if $(x^L : U_1) \in (\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{MN}$ and $(x^L : U_2) \in (\Gamma_1 \upharpoonright_M) \sqcap (\Gamma_2 \upharpoonright_N)$ then $U_1 \sqsubseteq U_2$:

– If $x^L \in \mathrm{fv}(M) \cap \mathrm{fv}(N)$ then $(x^L : U_1') \in \Gamma_1$, $(x^L : U_1'') \in \Gamma_2$ and $U_1 = U_1' \sqcap U_1'' = U_2$.
– If $x^L \in \mathrm{fv}(M) \setminus \mathrm{fv}(N)$ then
  - If $x^L \in \mathrm{dom}(\Gamma_2)$ then $(x^L : U_2) \in \Gamma_1$, $(x^L : U_1') \in \Gamma_2$ and $U_1 = U_1' \sqcap U_2 \sqsubseteq U_2$.
  - If $x^L \notin \mathrm{dom}(\Gamma_2)$ then $(x^L : U_2) \in \Gamma_1$ and $U_1 = U_2$.
– If $x^L \in \mathrm{fv}(N) \setminus \mathrm{fv}(M)$ then
  - If $x^L \in \mathrm{dom}(\Gamma_1)$ then $(x^L : U_1') \in \Gamma_1$, $(x^L : U_2) \in \Gamma_2$ and $U_1 = U_1' \sqcap U_2 \sqsubseteq U_2$.
  - If $x^L \notin \mathrm{dom}(\Gamma_1)$ then $x^L : U_2 \in \Gamma_2$ and $U_1 = U_2$.

3. Let $\Gamma = (x_j^{L_j} : U_j)_n$ and let $\mathrm{fv}(M) = \{y_1^{K_1}, \ldots, y_m^{K_m}\}$ where $m \leq n$ and $\forall 1 \leq k \leq m \ \exists 1 \leq j \leq n$ such that $y_k^{K_k} = x_j^{L_j}$. So $\Gamma \upharpoonright_M = (y_k^{K_k} : U_k)_m$ and $\overline{e}_i(\Gamma \upharpoonright_M) = (y_k^{i::K_k} : \overline{e}_i U_k)_m$. Since $\overline{e}_i \Gamma = (x_j^{i::L_j} : \overline{e}_i U_j)_n$, $\mathrm{fv}(M^{+i}) = \{y_1^{i::K_1}, \ldots, y_m^{i::K_m}\}$ and $\forall 1 \leq k \leq m \ \exists 1 \leq j \leq n$ such that $y_k^{i::K_k} = x_j^{i::L_j}$ then $(\overline{e}_i \Gamma) \upharpoonright_{M^{+i}} = (y_k^{i::K_k} : U_k)_m$. □

The next two theorems are needed in the proof of subject reduction.

Theorem 7. *If $M : \langle \Gamma \vdash U \rangle$ and $M \rhd_\beta N$, then $N : \langle \Gamma \upharpoonright_N \vdash U \rangle$.*

Proof. By induction on the derivation $M : \langle \Gamma \vdash U \rangle$.

– Rule ω follows by theorem 1.2 and lemma 25.1.
– If $\dfrac{M : \langle \Gamma, (x^L : U) \vdash T \rangle}{\lambda x^L.M : \langle \Gamma \vdash U \to T \rangle}$ then $N = \lambda x^L.N'$ and $M \rhd_\beta N'$. By IH, $N' : \langle (\Gamma, (x^L : U)) \upharpoonright_{N'} \vdash T \rangle$. If $x^L \in \mathrm{fv}(N')$ then $N' : \langle \Gamma \upharpoonright_{\mathrm{fv}(N') \setminus \{x^L\}}, (x^L : U) \vdash T \rangle$ and by $\to_I$, $\lambda x^L.N' : \langle \Gamma \upharpoonright_{\lambda x^L.N'} \vdash U \to T \rangle$. Else $N' : \langle \Gamma \upharpoonright_{\mathrm{fv}(N') \setminus \{x^L\}} \vdash T \rangle$ so by $\to_I'$, $\lambda x^L.N' : \langle \Gamma \upharpoonright_{\lambda x^L.N'} \vdash \omega^L \to T \rangle$ and since by lemma 2.4, $U \sqsubseteq \omega^L$, by $\sqsubseteq$, $\lambda x^L.N' : \langle \Gamma \upharpoonright_{\lambda x^L.N'} \vdash U \to T \rangle$.
– If $\dfrac{M : \langle \Gamma \vdash T \rangle \quad x^L \notin \mathrm{dom}(\Gamma)}{\lambda x^L.M : \langle \Gamma \vdash \omega^L \to T \rangle}$ then $N = \lambda x^L N'$ and $M \rhd_\beta N'$. Since $x^L \notin \mathrm{fv}(M)$, by theorem 1.2, $x^L \notin \mathrm{fv}(N')$. By IH, $N' : \langle \Gamma \upharpoonright_{\mathrm{fv}(N') \setminus \{x^L\}} \vdash T \rangle$ so by $\to_I'$, $\lambda x^L.N' : \langle \Gamma \upharpoonright_{\lambda x^L.N'} \vdash \omega^L \to T \rangle$.
– If $\dfrac{M_1 : \langle \Gamma_1 \vdash U \to T \rangle \quad M_2 : \langle \Gamma_2 \vdash U \rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1\ M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T \rangle}$. Using lemma 25.2, case $M_1 \rhd_\beta N_1$ and $N = N_1 M_2$ and case $M_2 \rhd_\beta N_2$ and $N = M_1 N_2$ are easy. Let $M_1 = \lambda x^L.M_1'$ and $N = M_1'[x^L := M_2]$. By lemma 4.3 and lemma 18.3, $M_1' \diamond M_2$. If $x^L \in FV(M_1')$ then by lemma 5.2, $M_1' : \langle \Gamma_1, x^L : U \vdash T \rangle$. By lemma 6, $M_1'[x^L := M_2] : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T \rangle$. If $x^L \notin FV(M_1')$ then by lemma 5.3, $M_1'[x^L := M_2] = M_1' : \langle \Gamma_1 \vdash T \rangle$ and by $\sqsubseteq$, $N : \langle (\Gamma_1 \sqcap \Gamma_2) \upharpoonright_N \vdash T \rangle$.
– Case $\sqcap_I$ is by IH.
– If $\dfrac{M : \langle \Gamma \vdash U \rangle}{M^{+i} : \langle \overline{e}_i \Gamma \vdash \overline{e}_i U \rangle}$ and $M^{+i} \rhd_\beta N$, then by lemma 19.10, there is $P \in \mathcal{M}$ such that $P^{+i} = N$ and $M \rhd_\beta P$. By IH, $P : \langle \Gamma \upharpoonright_P \vdash U \rangle$ and by e and lemma 25.3, $N : \langle (\overline{e}_i \Gamma) \upharpoonright_N \vdash \overline{e}_i U \rangle$.

– If $\dfrac{M : \langle \Gamma \vdash U\rangle \qquad \langle \Gamma \vdash U\rangle \sqsubseteq \langle \Gamma' \vdash U'\rangle}{M : \langle \Gamma' \vdash U'\rangle}$ then by IH, lemma 3.4 and $\sqsubseteq$, $N : \langle \Gamma' \upharpoonright_N \vdash U'\rangle$. □

Theorem 8. *If $M : \langle \Gamma \vdash U\rangle$ and $M \rhd_\eta N$, then $N : \langle \Gamma \vdash U\rangle$.*

Proof. By induction on the derivation $M : \langle \Gamma \vdash U\rangle$.

– If $\dfrac{}{M : \langle env^\omega_M \vdash \omega^{\mathrm{d}(M)}\rangle}$ then by lemma 1.1, $\mathrm{d}(M) = \mathrm{d}(N)$ and $\mathrm{fv}(M) = \mathrm{fv}(N)$ and by ω, $N : \langle env^\omega_M \vdash \omega^{\mathrm{d}(M)}\rangle$.
– If $\dfrac{M : \langle \Gamma, (x^L : U) \vdash T\rangle}{\lambda x^L.M : \langle \Gamma \vdash U \to T\rangle}$ then we have two cases:
  - $M = Nx^L$ and so by lemma 5.4, $N : \langle \Gamma \vdash U \to T\rangle$.
  - $N = \lambda x^L.N'$ and $M \rhd_\eta N'$. By IH, $N' : \langle \Gamma, (x^L : U) \vdash T\rangle$ and by $\to_I$, $N : \langle \Gamma \vdash U \to T\rangle$.
– if $\dfrac{M : \langle \Gamma \vdash T\rangle \quad x^L \notin \mathrm{dom}(\Gamma)}{\lambda x^L.M : \langle \Gamma \vdash \omega^L \to T\rangle}$ then $N = \lambda x^L.N'$ and $M \rhd_\eta N'$. By IH, $N' : \langle \Gamma \vdash T\rangle$ and by $\to'_I$, $N : \langle \Gamma \vdash \omega^L \to T\rangle$.
– If $\dfrac{M_1 : \langle \Gamma_1 \vdash U \to T\rangle \quad M_2 : \langle \Gamma_2 \vdash U\rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle}$, then we have two cases:
  - $M_1 \rhd_\eta N_1$ and $N = N_1M_2$. By IH $N_1 : \langle \Gamma_1 \vdash U \to T\rangle$ and by $\to_E$, $N : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle$.
  - $M_2 \rhd_\eta N_2$ and $N = M_1N_2$. By IH $N_2 : \langle \Gamma_2 \vdash U\rangle$ and by $\to_E$, $N : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle$.
– Case $\sqcap_I$ is by IH and $\sqcap_I$.
– If $\dfrac{M : \langle \Gamma \vdash U\rangle}{M^{+i} : \langle \overline{e}_i\Gamma \vdash \overline{e}_iU\rangle}$ then by lemma 19.10, there is $P \in \mathcal{M}$ such that $P^{+i} = N$ and $M \rhd_\eta P$. By IH, $P : \langle \Gamma \vdash U\rangle$ and by e, $N : \langle \overline{e}_i\Gamma \vdash \overline{e}_iU\rangle$.
– If $\dfrac{M : \langle \Gamma \vdash U\rangle \qquad \langle \Gamma \vdash U\rangle \sqsubseteq \langle \Gamma' \vdash U'\rangle}{M : \langle \Gamma' \vdash U'\rangle}$ then by IH, lemma 3.4 and $\sqsubseteq$, $N : \langle \Gamma' \vdash U'\rangle$. □

The next auxiliary lemma is needed in proofs.

Lemma 26. *Let $i \in \{1, 2\}$ and $M : \langle \Gamma \vdash U\rangle$. We have:*

1. *If $(x^L : U_1) \in \Gamma$ and $(y^K : U_2) \in \Gamma$, then:*
   (a) If $(x^L : U_1) \neq (y^K : U_2)$, then $x^L \neq y^K$.
   (b) If $x = y$, then $L = K$ and $U_1 = U_2$.
2. *If $(x^L : U_1) \in \Gamma$ and $(y^K : U_2) \in \Gamma$ and $(x^L : U_1) \neq (y^K : U_2)$, then $x \neq y$ and $x^L \neq y^K$.*

Proof. 1. If $x^L = Y^K$ then by definition $U_1 = U_2$, so $(x^L : U_1) = (y^K : U_2)$. By lemma 4.2, $x^L, y^K \in \mathrm{fv}(M)$. By lemma 18.1, $M \diamond M$. So, if $x = y$ then $L = K$ and by definition $U_1 = U_2$. 2. Corollary of 1. □

Proof (Of theorem 4). Proofs are by induction on derivations using theorem 7 and theorem 8. □

D Proofs for section 5

Proof (Of lemma 7). By lemma 3.2, $M[x^L := N] \in \mathcal{M}$, so by definition, $M, N \in \mathcal{M}$ and $M \diamond N$ and $\mathrm{d}(N) = L$. By induction on the derivation $M[x^L := N] : \langle \Gamma \vdash U\rangle$.

- If $\dfrac{}{y^{\oslash} : \langle (y^{\oslash} : T) \vdash T\rangle}$ then $M = x^{\oslash}$ and $N = y^{\oslash}$. By ax, $x^{\oslash} : \langle (x^{\oslash} : T) \vdash T\rangle$.
- If $\dfrac{}{M[x^L := N] : \langle env^{\omega}_{M[x^L := N]} \vdash \omega^{\mathrm{d}(M[x^L := N])}\rangle}$ then by lemma 18.5, $\mathrm{d}(M) = \mathrm{d}(M[x^L := N])$. By ω, $M : \langle env^{\omega}_{\mathrm{fv}(M)\setminus\{x^L\}}, (x^L : \omega^L) \vdash \omega^{\mathrm{d}(M)}\rangle$ and $N : \langle env^{\omega}_N \vdash \omega^L\rangle$ and because $\mathrm{fv}(M[x^L := N]) = (\mathrm{fv}(M) \setminus \{x^L\}) \cup \mathrm{fv}(N)$, $env^{\omega}_{\mathrm{fv}(M)\setminus\{x^L\}} \sqcap env^{\omega}_N = env^{\omega}_{M[x^L := N]}$.
- If $\dfrac{M[x^L := N] : \langle \Gamma, (y^K : W) \vdash T\rangle}{\lambda y^K.M[x^L := N] : \langle \Gamma \vdash W \to T\rangle}$ where $y^K \notin \mathrm{fv}(N) \cup \{x^L\}$. By IH, $\exists\, V$ and $\exists\, \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma_1, x^L : V \vdash T\rangle$, $N : \langle \Gamma_2 \vdash V\rangle$ and $\Gamma, y^K : W = \Gamma_1 \sqcap \Gamma_2$. By lemma 4.2, $\mathrm{fv}(N) = \mathrm{dom}(\Gamma_2)$ and $\mathrm{fv}(M) = \mathrm{dom}(\Gamma_1) \cup \{y^K\}$. Since $y^K \in \mathrm{fv}(M)$ and $y^K \notin \mathrm{fv}(N)$, $\Gamma_1 = \Delta_1, y^K : W$. Hence $M : \langle \Delta_1, y^K : W, x^L : V \vdash T\rangle$. By rule $\to_I$, $\lambda y^K.M : \langle \Delta_1, x^L : V \vdash W \to T\rangle$. Finally $\Gamma = \Delta_1 \sqcap \Gamma_2$.
- If $\dfrac{M[x^L := N] : \langle \Gamma \vdash T\rangle \quad y^K \notin \mathrm{dom}(\Gamma)}{\lambda y^K.M[x^L := N] : \langle \Gamma \vdash \omega^K \to T\rangle}$ where $y^K \notin \mathrm{fv}(N) \cup \{x^L\}$. By IH, $\exists\, V$ type and $\exists\, \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma_1, x^L : V \vdash T\rangle$, $N : \langle \Gamma_2 \vdash V\rangle$ and $\Gamma = \Gamma_1 \sqcap \Gamma_2$. Since $y^K \neq x^L$, $\lambda y^K.M : \langle \Gamma_1, x^L : V \vdash \omega^K \to T\rangle$.
- If $\dfrac{M_1[x^L := N] : \langle \Gamma_1 \vdash W \to T\rangle \quad M_2[x^L := N] : \langle \Gamma_2 \vdash W\rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1[x^L := N]\ M_2[x^L := N] : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle}$ where $M = M_1 M_2$, then we have three cases:
  - If $x^L \in \mathrm{fv}(M_1) \cap \mathrm{fv}(M_2)$ then by IH, $\exists\, V_1, V_2$ types and $\exists\, \Delta_1, \Delta_2, \nabla_1, \nabla_2$ type environments such that $M_1 : \langle \Delta_1, (x^L : V_1) \vdash W \to T\rangle$, $M_2 : \langle \nabla_1, (x^L : V_2) \vdash W\rangle$, $N : \langle \Delta_2 \vdash V_1\rangle$, $N : \langle \nabla_2 \vdash V_2\rangle$, $\Gamma_1 = \Delta_1 \sqcap \Delta_2$ and $\Gamma_2 = \nabla_1 \sqcap \nabla_2$. Because $\Gamma_1 \diamond \Gamma_2$, then $\Delta_1 \diamond \nabla_1$ and $\Delta_2 \diamond \nabla_2$ and because $\Delta_1, (x^L : V_1)$ and $\nabla_1, (x^L : V_2)$ are type environments, by lemma 26, $(\Delta_1, (x^L : V_1)) \diamond (\nabla_1, (x^L : V_2))$. Then, by rules $\sqcap_I$ and $\to_E$, $M_1 M_2 : \langle \Delta_1 \sqcap \nabla_1, (x^L : V_1 \sqcap V_2) \vdash T\rangle$ and by $\sqcap'_I$, $N : \langle \Delta_2 \sqcap \nabla_2 \vdash V_1 \sqcap V_2\rangle$. Finally, $\Gamma_1 \sqcap \Gamma_2 = (\Delta_1 \sqcap \Delta_2) \sqcap (\nabla_1 \sqcap \nabla_2)$.
  - If $x^L \in \mathrm{fv}(M_1) \setminus \mathrm{fv}(M_2)$ then by IH, $\exists\, V$ types and $\exists\, \Delta_1, \Delta_1$ type environments such that $M_1 : \langle \Delta_1, (x^L : V) \vdash W \to T\rangle$, $N : \langle \Delta_2 \vdash V\rangle$ and $\Gamma_1 = \Delta_1 \sqcap \Delta_2$. Since $\Gamma_1 \diamond \Gamma_2$, $\Delta_1 \diamond \Gamma_2$ and since $\Gamma_1 \sqcap \Gamma_2$ is a type environment, by lemma 26, $(\Delta_1, (x^L : V)) \diamond \Gamma_2$. By $\to_E$, $M_1 M_2 : \langle \Delta_1 \sqcap \Gamma_2, (x^L : V) \vdash T\rangle$ and $\Gamma_1 \sqcap \Gamma_2 = (\Delta_1 \sqcap \Delta_2) \sqcap \Gamma_2$.
  - If $x^L \in \mathrm{fv}(M_2) \setminus \mathrm{fv}(M_1)$ then by IH, $\exists\, V$ types and $\exists\, \Delta_1, \Delta_2$ type environments such that $M_2 : \langle \Delta_1, (x^L : V) \vdash W\rangle$, $N : \langle \Delta_2 \vdash V\rangle$ and $\Gamma_2 = \Delta_1 \sqcap \Delta_2$. Since $\Gamma_1 \diamond \Gamma_2$, $\Gamma_1 \diamond \Delta_1$ and since $\Gamma_1 \sqcap \Gamma_2$ is a type environment, by

lemma 26, $(\Delta_1, (x^L : V)) \diamond \Gamma_1$. By $\rightarrow_E$, $M_1 M_2 : \langle \Gamma_1 \sqcap \Delta_1, (x^L : V) \vdash T\rangle$ and $\Gamma_1 \sqcap \Gamma_2 = \Gamma_1 \sqcap (\Delta_1 \sqcap \Delta_2)$.

– Let $\dfrac{M[x^L := N] : \langle \Gamma \vdash U_1\rangle \quad M[x^L := N] : \langle \Gamma \vdash U_2\rangle}{M[x^L := N] : \langle \Gamma \vdash U_1 \sqcap U_2\rangle}$. By IH, $\exists\ V_1, V_2$ types and $\exists\ \Delta_1, \Delta_2, \nabla_1, \nabla_2$ type environments such that $M : \langle \Delta_1, x^L : V_1 \vdash U_1\rangle$, $M : \langle \nabla_1, x^L : V_2 \vdash U_2\rangle$, $N : \langle \Delta_2 \vdash V_1\rangle$, $N : \langle \nabla_2 \vdash V_2\rangle$, $\Gamma = \Delta_1 \sqcap \Delta_2$ and $\Gamma = \nabla_1 \sqcap \nabla_2$. Then, by rule $\sqcap'_I$, $M : \langle \Delta_1 \sqcap \nabla_1, x^L : V_1 \sqcap V_2 \vdash U_1 \sqcap U_2\rangle$ and $N : \langle \Delta_2 \sqcap \nabla_2 \vdash V_1 \sqcap V_2\rangle$. Finally, $\Gamma = (\Delta_1 \sqcap \Delta_2) \sqcap (\nabla_1 \sqcap \nabla_2)$.
– If $\dfrac{M[x^L := N] : \langle \Gamma \vdash U\rangle}{M^{+j}[x^{j::L} := N^{+j}] : \langle \overline{e}_j \Gamma \vdash \overline{e}_j U\rangle}$ then by IH, $\exists\ V$ type and $\exists\ \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma_1, x^L : V \vdash U\rangle$, $N : \langle \Gamma_2 \vdash V\rangle$ and $\Gamma = \Gamma_1 \sqcap \Gamma_2$. So by e, $M^{+j} : \langle \overline{e}_j \Gamma_1, x^{j::L} : \overline{e}_j V \vdash \overline{e}_j U\rangle$, $N : \langle \overline{e}_j \Gamma_2 \vdash \overline{e}_j V\rangle$ and $\overline{e}_j \Gamma = \overline{e}_j \Gamma_1 \sqcap \overline{e}_j \Gamma_2$.
– If $\dfrac{M[x^L := N] : \langle \Gamma' \vdash U'\rangle \quad \langle \Gamma' \vdash U'\rangle \sqsubseteq \langle \Gamma \vdash U\rangle}{M[x^L := N] : \langle \Gamma \vdash U\rangle}$ then by lemma 3.3, $\Gamma \sqsubseteq \Gamma'$ and $U' \sqsubseteq U$. By IH, $\exists\ V$ type and $\exists\ \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma'_1, x^L : V \vdash U'\rangle$, $N : \langle \Gamma'_2 \vdash V\rangle$ and $\Gamma' = \Gamma'_1 \sqcap \Gamma'_2$. Then by lemma 2.6, $\Gamma = \Gamma_1 \sqcap \Gamma_2$ and $\Gamma_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$. So by $\sqsubseteq$, $M : \langle \Gamma_1, x^L : V \vdash U\rangle$ and $N : \langle \Gamma_2 \vdash V\rangle$. □

The next lemma is basic for the proof of subject expansion for β.

Lemma 27. *If $M[x^L := N] : \langle \Gamma \vdash U\rangle$, $d(U) = K$ and $L \succeq d(M)$, $\mathcal{U} = \mathrm{fv}((\lambda x^L.M)N)$, then $(\lambda x^L.M)N : \langle \Gamma{\uparrow}^{\mathcal{U}} \vdash U\rangle$.*

Proof. By lemma 3.2, $M[x^L := N] \in \mathcal{M}$, so $M, N \in \mathcal{M}$ and $M \diamond N$ and $\mathrm{d}(N) = L$. By definition $(\lambda x^L.M)N \in \mathcal{M}$. By lemma 18.5 and theorem 3.2, $\mathrm{d}(\Gamma) \succeq \mathrm{d}(U) = K = \mathrm{d}(M[x^L := N]) = \mathrm{d}(M) = \mathrm{d}((\lambda x^L.M)N)$. So $L \succeq K$ and there exists K' such that $L = K :: K'$. We have two cases:

– If $x^L \in \mathrm{fv}(M)$, then, by lemma 7, $\exists\ V$ type and $\exists\ \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma_1, x^L : V \vdash U\rangle$, $N : \langle \Gamma_2 \vdash V\rangle$ and $\Gamma = \Gamma_1 \sqcap \Gamma_2$. By lemma 3.2, $\mathrm{OK}(\Gamma_1)$ and $\mathrm{OK}(\Gamma_2)$. By lemma 3.9, $\mathrm{OK}(\Gamma_1 \sqcap \Gamma_2)$. So, it is easy to prove, using lemma 3.1, that $\mathrm{OK}(\Gamma{\uparrow}^{\mathcal{U}})$. By lemma 4.3, $\Gamma_1, x^L : V \diamond \Gamma_2$, so $\Gamma_1 \diamond \Gamma_2$. By lemma 3.2, $\mathrm{d}(\Gamma_1) \succeq \mathrm{d}(M) = \mathrm{d}(U) = K$ and $L = \mathrm{d}(N) = \mathrm{d}(V) \preceq \mathrm{d}(\Gamma_2)$. By lemma 2, we have two cases :
  - If $U = \omega^K$, then by lemma 4.1, $(\lambda x^L.M)N : \langle \Gamma{\uparrow}^{\mathcal{U}} \vdash U\rangle$.
  - If $U = \boldsymbol{e}_K \sqcap_{i=1}^p T_i$ where $p \geq 1$ and $\forall\ 1 \leq i \leq p$, $T_i \in \mathbb{T}$, then by theorem 3.3, $M^{-K} : \langle \Gamma_1^{-K}, x^{K'} : V^{-K} \vdash \sqcap_{i=1}^p T_i\rangle$. By $\sqsubseteq$, $\forall\ 1 \leq i \leq p$, $M^{-K} : \langle \Gamma_1^{-K}, x^{K'} : V^{-K} \vdash T_i\rangle$, so by $\rightarrow_I$, $\lambda x^{K'}.M^{-K} : \langle \Gamma_1^{-K} \vdash V^{-K} \rightarrow T_i\rangle$. Again by theorem 3.3, $N^{-K} : \langle \Gamma_2^{-K} \vdash V^{-K}\rangle$ and since $\Gamma_1 \diamond \Gamma_2$, by lemma 3.6, $\Gamma_1^{-K} \diamond \Gamma_2^{-K}$, so by $\rightarrow_E$, $\forall\ 1 \leq i \leq p$, $(\lambda x^{K'}.M^{-K})N^{-K} : \langle \Gamma_1^{-K} \sqcap \Gamma_2^{-K} \vdash T_i\rangle$. Finally by $\sqcap_I$ and e, $(\lambda x^L.M)N : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U\rangle$, so $(\lambda x^L.M)N : \langle \Gamma{\uparrow}^{\mathcal{U}} \vdash U\rangle$.

– If $x^L \notin \mathrm{fv}(M)$, then $M : \langle \Gamma \vdash U\rangle$. By lemma 3.2, $\mathrm{OK}(\Gamma)$. So, it is easy to prove, using lemma 3.1, that $\mathrm{OK}(\Gamma{\uparrow}^{\mathcal{U}})$. By lemma 2, we have two cases :

- If $U = \omega^K$, then by lemma 4.1, $(\lambda x^L.M)N : \langle \Gamma\uparrow^{\mathcal{U}} \vdash U\rangle$.
- If $U = \boldsymbol{e}_K \sqcap_{i=1}^p T_i$ where $p \geq 1$ and $\forall\ 1 \leq i \leq p$, $T_i \in \mathbb{T}$, then by theorem 3.3, $M^{-K} : \langle \Gamma^{-K} \vdash \sqcap_{i=1}^p T_i\rangle$. By $\sqsubseteq$, $\forall\ 1 \leq i \leq p$, $M^{-K} : \langle \Gamma^{-K} \vdash T_i\rangle$. Using lemma 19 and by induction on K, we can prove that $x^{K'} \notin \mathrm{fv}(M^{-K})$. So by lemma 4.2, $x^{K'} \notin \mathrm{dom}(\Gamma^{-K})$. So by $\rightarrow'_I$, $\lambda x^{K'}.M^{-K} : \langle \Gamma^{-K} \vdash \omega^{K'} \rightarrow T_i\rangle$. By (ω), $N^{-K} : \langle env^{\omega}_{N^{-K}} \vdash \omega^{K'}\rangle$ and $N : \langle env^{\omega}_N \vdash \omega^L\rangle$. By theorem 3.2, $\mathrm{d}(env^{\omega}_N) \succeq \mathrm{d}(N) = L$. By lemma 4.3, $\Gamma \diamond env^{\omega}_N$. By lemma 3.6, $\Gamma^{-K} \diamond env^{\omega}_{N^{-K}}$. By $\rightarrow_E$, $\forall\ 1 \leq i \leq p$, $(\lambda x^{K'}.M^{-K})N^{-K} : \langle \Gamma^{-K} \sqcap env^{\omega}_{N^{-K}} \vdash T_i\rangle$. Finally by $\sqcap_I$ and e, $(\lambda x^L.M)N : \langle \Gamma \sqcap env^{\omega}_N \vdash U\rangle$, so $(\lambda x^L.M)N : \langle \Gamma\uparrow^{\mathcal{U}} \vdash U\rangle$. □

Next, we give the main block for the proof of subject expansion for β.

Theorem 9. *If $N : \langle \Gamma \vdash U\rangle$ and $M \rhd_\beta N$, then $M : \langle \Gamma\uparrow^M \vdash U\rangle$.*

Proof. By induction on the derivation $N : \langle \Gamma \vdash U\rangle$.

- If $\dfrac{}{x^{\oslash} : \langle (x^{\oslash} : T) \vdash T\rangle}$ and $M \rhd_\beta x^{\oslash}$, then $M = (\lambda y^K.M_1)M_2$ and $x^{\oslash} = M_1[y^K := M_2]$. Because $M \in \mathcal{M}$ then $K \succeq \mathrm{d}(M_1)$. By lemma 27, $M : \langle (x^{\oslash} : T)\uparrow^M \vdash T\rangle$.
- If $\dfrac{}{N : \langle env^{\omega}_N \vdash \omega^{\mathrm{d}(N)}\rangle}$ and $M \rhd_\beta N$, then since by theorem 1.2, $\mathrm{fv}(N) \subseteq \mathrm{fv}(M)$ and $\mathrm{d}(M) = \mathrm{d}(N)$, $(env^{\omega}_N)\uparrow^M = env^{\omega}_M$. By ω, $M : \langle env^{\omega}_M \vdash \omega^{\mathrm{d}(M)}\rangle$. Hence, $M : \langle (env^N_{\omega})\uparrow^M \vdash \omega^{\mathrm{d}(N)}\rangle$.
- If $\dfrac{N : \langle \Gamma, x^L : U \vdash T\rangle}{\lambda x^L.N : \langle \Gamma \vdash U \rightarrow T\rangle}$ and $M \rhd_\beta \lambda x^L.N$, then we have two cases:
  - If $M = \lambda x.M'$ where $M' \rhd_\beta N$, then by IH, $M' : \langle (\Gamma, (x^L : U))\uparrow^{M'} \vdash T\rangle$. Since by theorem 1.2 and lemma 4.2, $x^L \in \mathrm{fv}(N) \subseteq \mathrm{fv}(M')$, then we have $(\Gamma, (x^L : U))\uparrow^{\mathrm{fv}(M')} = \Gamma\uparrow^{\mathrm{fv}(M')\setminus\{x^L\}}, (x^L : U)$ and $\Gamma\uparrow^{\mathrm{fv}(M')\setminus\{x^L\}} = \Gamma\uparrow^{\lambda x^L.M'}$. Hence, $M' : \langle \Gamma\uparrow^{\lambda x^L.M'}, (x^L : U) \vdash T\rangle$ and finally, by $\rightarrow_I$, $\lambda x^L.M' : \langle \Gamma\uparrow^{\lambda x^L.M'} \vdash U \rightarrow T\rangle$.
  - If $M = (\lambda y^K.M_1)M_2$ where $y^K \notin \mathrm{fv}(M_2)$ and $\lambda x^L.N = M_1[y^K := M_2]$, then, because $M \in \mathcal{M}$ then $K \succeq \mathrm{d}(M_1)$ and by lemma 27, Because $M_1[y^K := M_2] : \langle \Gamma \vdash U \rightarrow T\rangle$, we have $(\lambda y^K.M_1)M_2 : \langle \Gamma\uparrow^{(\lambda y^K.M_1)M_2} \vdash U \rightarrow T\rangle$.
- If $\dfrac{N : \langle \Gamma \vdash T\rangle \quad x^L \notin \mathrm{dom}(\Gamma)}{\lambda x^L.N : \langle \Gamma \vdash \omega^L \rightarrow T\rangle}$ and $M \rhd_\beta N$ then similar to the above case.
- If $\dfrac{N_1 : \langle \Gamma_1 \vdash U \rightarrow T\rangle \quad N_2 : \langle \Gamma_2 \vdash U\rangle \quad \Gamma_1 \diamond \Gamma_2}{N_1\ N_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle}$ and $M \rhd_\beta N_1N_2$, we have three cases:
  - $M = M_1N_2$ where $M_1 \rhd_\beta N_1$ and $M_1 \diamond N_2$. By IH, $M_1 : \langle \Gamma_1\uparrow^{M_1} \vdash U \rightarrow T\rangle$. It is easy to show that $(\Gamma_1 \sqcap \Gamma_2)\uparrow^{M_1N_2} = \Gamma_1\uparrow^{M_1} \sqcap \Gamma_2$. Since $M_1 \diamond N_2$, by lemma 4.3, $\Gamma_1\uparrow^{M_1} \diamond \Gamma_2$, hence use $\rightarrow_E$.
  - $M = N_1M_2$ where $M_2 \rhd_\beta N_2$. Similar to the above case.

  - If $M = (\lambda x^L.M_1)M_2$ and $N_1 N_2 = M_1[x^L := M_2]$ then, because $M \in \mathcal{M}$ then $L \succeq \mathrm{d}(M_1)$ and by lemma 27, $(\lambda x^L.M_1)M_2 : \langle (\Gamma_1 \sqcap \Gamma_2)\uparrow^{(\lambda x^L.M_1)M_2} \vdash T\rangle$.

– If $\dfrac{N : \langle \Gamma \vdash U_1\rangle \quad N : \langle \Gamma \vdash U_2\rangle}{N : \langle \Gamma \vdash U_1 \sqcap U_2\rangle}$ and $M \rhd_\beta N$ then use IH.

– If $\dfrac{N : \langle \Gamma \vdash U\rangle}{N^{+j} : \langle \overline{e}_j \Gamma \vdash \overline{e}_j U\rangle}$ then by lemma 19.9 then there is $P \in \mathcal{M}$ such that $M = P^{+j}$ and $P \rhd_\beta N$. By IH, $P : \langle \Gamma\uparrow^P \vdash U\rangle$ and by e, $M : \langle (\overline{e}_j \Gamma)\uparrow^M \vdash \overline{e}_j U\rangle$.

– If $\dfrac{N : \langle \Gamma \vdash U\rangle \quad \langle \Gamma \vdash U\rangle \sqsubseteq \langle \Gamma' \vdash U'\rangle}{N : \langle \Gamma' \vdash U'\rangle}$ and $M \rhd_\beta N$. By lemma 3.4, $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$. It is easy to show that $\Gamma'\uparrow^M \sqsubseteq \Gamma\uparrow^M$ and hence by lemma 3.4, $\langle \Gamma\uparrow^M \vdash U\rangle \sqsubseteq \langle \Gamma'\uparrow^M \vdash U'\rangle$. By IH, $M\uparrow^M : \langle \Gamma \vdash U\rangle$. Hence, by $\sqsubseteq_{\langle\rangle}$, we have $M : \langle \Gamma'\uparrow^M \vdash U'\rangle$. □

Proof (Of theorem 5). By induction on the length of the derivation $M \rhd_\beta^* N$ using theorem 9 and the fact that if $\mathrm{fv}(P) \subseteq \mathrm{fv}(Q)$, then $(\Gamma\uparrow^P)\uparrow^Q = \Gamma\uparrow^Q$. □

E Proofs of section 6

Proof (Of lemma 9). 1. and 2. are easy.
3. If $M \rhd_r^* N^{+i}$ where $N \in \mathcal{X}$, then, by lemma 19.9, $M = P^{+i}$ such that $P \in \mathcal{M}$ and $P \rhd_r N$. As $\mathcal{X}$ is r-saturated, $P \in \mathcal{X}$ and so $P^{+i} = M \in \mathcal{X}^{+i}$.
4. Let $M \in \mathcal{X} \rightsquigarrow \mathcal{Y}$ and $N \rhd_r^* M$. If $P \in \mathcal{X}$ such that $P \diamond N$, then by lemma 19.8, $P \diamond M$. So, by definition, $MP \in \mathcal{Y}$. Because $\mathcal{Y} \subseteq \mathcal{M}$, then $MP \in \mathcal{M}$. Hence, $\mathrm{d}(M) \preceq \mathrm{d}(P)$. By lemma 1, $\mathrm{d}(M) = \mathrm{d}(N)$. So $NP \in \mathcal{M}$ and $NP \rhd_r^* MP$. Because $MP \in \mathcal{Y}$ and $\mathcal{Y}$ is r-saturated, then $NP \in \mathcal{Y}$. Hence, $N \in \mathcal{X} \rightsquigarrow \mathcal{Y}$.
5. Let $M \in (\mathcal{X} \rightsquigarrow \mathcal{Y})^{+i}$, then $M = N^{+i}$ and $N \in \mathcal{X} \rightsquigarrow \mathcal{Y}$. Let $P \in \mathcal{X}^{+i}$ such that $M \diamond P$. Then $P = Q^{+i}$ such that $Q \in \mathcal{X}$. Because $M \diamond P$ then by lemma 19.2, $N \diamond Q$. So $NQ \in \mathcal{Y}$. Because $\mathcal{Y} \subseteq \mathcal{M}$ then $NQ \in \mathcal{M}$. Because $(NQ)^{+i} = N^{+i}Q^{+i} = MP$ then $MP \in \mathcal{Y}^{+i}$. Hence, $M \in \mathcal{X}^{+i} \rightsquigarrow \mathcal{Y}^{+i}$.
6. Let $M \in \mathcal{X}^{+i} \rightsquigarrow \mathcal{Y}^{+i}$ such that $\mathcal{X}^{+i} \wr \mathcal{Y}^{+i}$. By hypothesis, there exists $P \in \mathcal{X}^{+i}$ such that $M \diamond P$. Then $MP \in \mathcal{Y}^{+i}$. Hence $MP = Q^{+i}$ such that $Q \in \mathcal{Y}$. Because $\mathcal{Y} \subseteq \mathcal{M}$ then $Q \in \mathcal{M}$ and by lemma 19.1, $MP \in \mathcal{M}$. Hence by definition $M \in \mathcal{M}$ and by lemma 19.1, $\mathrm{d}(M) = \mathrm{d}(Q^{+i}) = i :: \mathrm{d}(Q)$. So by lemma 19.7, there exists $M_1 \in \mathcal{M}$ such that $M = M_1^{+i}$. Let $N_1 \in \mathcal{X}$ such that $M_1 \diamond N_1$. By definition $N_1^{+i} \in \mathcal{X}^{+i}$ and by lemma 19.2, $M \diamond N_1^{+}$. So, $MN_1^{+i} \in \mathcal{Y}^{+i}$. So $MN_1^{+i} = M'^{+i}$ such that $M' \in \mathcal{Y}$. Because $\mathcal{Y} \subseteq \mathcal{M}$ then $M' \in \mathcal{M}$. By lemma 19.1, $MN_1^{+i} \in \mathcal{M}$. So $M_1^{+i} \diamond N_1^{+i}$ and $\mathrm{d}(M_1^{+i}) \preceq \mathrm{d}(N_1^{+i})$. By lemma 19.1 and lemma 19.2, $M_1 \diamond N_1$ and $\mathrm{d}(M_1) \preceq \mathrm{d}(N_1)$. So $M_1 N_1 \in \mathcal{M}$ and $(M_1 N_1)^{+i} = M_1^{+i} N_1^{+i} \in \mathcal{Y}^{+i}$. Hence $M_1 N_1 \in \mathcal{Y}$. Thus, $M_1 \in \mathcal{X} \rightsquigarrow \mathcal{Y}$ and $M = M_1^{+i} \in (\mathcal{X} \rightsquigarrow \mathcal{Y})^{+i}$. □

Proof (Of lemma 10). 1.1a . By induction on U using lemma 9 and lemma 1.
1.1b. We prove $\forall x \in \mathcal{V}_1, \mathcal{N}_x^L \subseteq \mathcal{I}(U) \subseteq \mathcal{M}^L$ by induction on U. Case $U = a$: by definition. Case $U = \omega^L$: We have $\forall x \in \mathcal{V}_1, \mathcal{N}_x^L \subseteq \mathcal{M}^L \subseteq \mathcal{M}^L$. Case $U =$

$U_1 \sqcap U_2$ (resp. $U = \bar{e}_i V$): use IH since $\mathrm{d}(U_1) = \mathrm{d}(U_2)$ (resp. $\mathrm{d}(U) = i :: \mathrm{d}(V)$, $\forall x \in \mathcal{V}_1, (\mathcal{N}_x^K)^{+i} = \mathcal{N}_x^{i::K}$ and $(\mathcal{M}^K)^{+i} = \mathcal{M}^{i::K}$). Case $U = V \to T$: by definition, $K = \mathrm{d}(V) \succeq \mathrm{d}(T) = \oslash$.

- Let $x \in \mathcal{V}_1, N_1, ..., N_k$ such that $\forall\, 1 \le i \le k, \mathrm{d}(N_i) \succeq \oslash$ and $\diamond\{x^\oslash, N_1, \ldots, N_k\}$ and let $N \in \mathcal{I}(V)$ such that $(x^\oslash N_1 ... N_k) \diamond N$. By IH, $\mathrm{d}(N) = K \succeq \oslash$. Again, by IH, $x^\oslash N_1 ... N_k N \in \mathcal{I}(T)$. Thus $x^\oslash N_1 ... N_k \in \mathcal{I}(V \to T)$.
- Let $M \in \mathcal{I}(V \to T)$. Let $x \in \mathcal{V}_1$ such that $\forall L, x^L \notin \mathrm{fv}(M)$. By IH, $x^K \in \mathcal{I}(V)$, then $Mx^K \in \mathcal{I}(T)$ and, by IH, $\mathrm{d}(Mx^K) = \oslash$. Thus $\mathrm{d}(M) = \oslash$.

2. By induction of the derivation $U \sqsubseteq V$. □

Proof (Of lemma 11). By induction on the derivation $M : \langle (x_j^{L_j} : U_j)_n \vdash U \rangle$.

- If $\dfrac{}{x^\oslash : \langle (x^\oslash : T) \vdash T \rangle}$ and $N \in \mathcal{I}(T)$, then $x^\oslash[x^\oslash := N] = N \in \mathcal{I}(T)$.
- If $\dfrac{}{M : \langle env_M^\omega \vdash \omega^{\mathrm{d}(M)} \rangle}$. Let $env_M^\omega = (x_j^{L_j} : U_j)_n$ so $\mathrm{fv}(M) = \{x_1^{L_1}, ..., x_n^{L_n}\}$. Because, by lemma 3.2, for all $j \in \{1, \ldots, n\}$, $\mathrm{d}(U_j) = L_j$ by lemma 10.1, $\mathcal{I}(U_j) \subseteq \mathcal{M}^{L_j}$, hence, $\mathrm{d}(N_j) = L_j$. Because $M[(x_j^{L_j} := N_j)_n] \in \mathcal{M}$, then $\diamond\{M\} \cup \{N_i \,/\, i \in \{1, \ldots, n\}\}$. Then, by lemma 18.5, $\mathrm{d}(M[(x_j^{L_j} := N_j)_n]) = \mathrm{d}(M)$ and $M[(x_j^{L_j} := N_j)_n] \in \mathcal{M}^{\mathrm{d}(M)} = \mathcal{I}(\omega^{\mathrm{d}(M)})$.
- If $\dfrac{M : \langle (x_j^{L_j} : U_j)_n, (x^K : V) \vdash T \rangle}{\lambda x^K.M : \langle (x_j^{L_j} : U_j)_n \vdash V \to T \rangle}$, $\forall 1 \le j \le n$, $N_j \in \mathcal{I}(U_j)$ and $N \in \mathcal{I}(V)$ such that $(\lambda x^K.M)[(x_j^{L_j} := N_j)_n] \diamond N$. By lemma 3.2, $\mathrm{d}(V) = K$. We have, $(\lambda x^K.M)[(x_j^{L_j} := N_j)_n] = \lambda x^K.M[(x_j^{L_j} := N_j)_n]$, where $\forall 1 \le j \le n, y^K \notin \mathrm{fv}(N_j) \cup \{x_j^{L_j}\}$. Since $N \in \mathcal{I}(V)$ and by lemma 10.1, $\mathcal{I}(V) \subseteq \mathcal{M}^K$, $\mathrm{d}(N) = K$. By lemma 18.3 and lemma 18.5, $M[(x_j^{L_j} := N_j)_n] \diamond N$ and $M[(x_j^{L_j} := N_j)_n][x^K := N] = M[(x_j^{L_j} := N_j)_n, x^K := N] \in \mathcal{M}$. Hence, $(\lambda x^K.M[(x_j^{L_j} := N_j)_n])N \in \mathcal{M}$ and $(\lambda x^K.M[(x_j^{L_j} := N_j)_n])N \rhd_r M[(x_j^{L_j} := N_j)_n, (x^K := N)]$. By IH, $M[(x_j^{L_j} := N_j)_n, (x^K := N)] \in \mathcal{I}(T)$. Since, by lemma 10.1 $\mathcal{I}(T)$ is r-saturated, then $(\lambda x^K.M[(x_j^{L_j} := N_j)_n])N \in \mathcal{I}(T)$ and so $\lambda x^K.M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(V) \rightsquigarrow \mathcal{I}(T) = \mathcal{I}(V \to T)$.
- If $\dfrac{M : \langle (x_j^{L_j} : U_j)_n \vdash T \rangle \quad x^K \notin \mathrm{dom}((x_j^{L_j} : U_j)_n)}{\lambda x^K.M : \langle (x_j^{L_j} : U_j)_n \vdash \omega^K \to T \rangle}$, $\forall 1 \le j \le n$, $N_j \in \mathcal{I}(U_j)$ and $N \in \mathcal{I}(\omega^K)$ such that $(\lambda x^K.M)[(x_j^{L_j} := N_j)_n] \diamond N$. By lemma 4.2, $x^K \notin \mathrm{fv}(M)$. We have, $(\lambda x^K.M)[(x_j^{L_j} := N_j)_n] = \lambda x^K.M[(x_j^{L_j} := N_j)_n]$, where $\forall 1 \le j \le n, x^K \notin \mathrm{fv}(N_j) \cup \{x_j^{L_j}\}$. Since $N \in \mathcal{I}(\omega^K)$ and by lemma 10.1, $\mathcal{I}(\omega^K) = \mathcal{M}^K$ then $\mathrm{d}(N) = K$. By lemma 18.3 and lemma 18.5, $M[(x_j^{L_j} := N_j)_n] \diamond N$ and $M[(x_j^{L_j} := N_j)_n][x^K := N] = M[(x_j^{L_j} := N_j)_n, x^K := N] = M[(x_j^{L_j} := N_j)_n] \in \mathcal{M}$. Hence, $(\lambda x^K.M[(x_j^{L_j} := N_j)_n])N \in \mathcal{M}$

and $(\lambda x^K.M[(x_j^{L_j} := N_j)_n])N \rhd_r M[(x_j^{L_j} := N_j)_n, (x^K := N)]$. By IH, $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(T)$. Since, by lemma 10.1 $\mathcal{I}(T)$ is r-saturated, then $(\lambda x^K.M[(x_j^{L_j} := N_j)_n])N \in \mathcal{I}(T)$ and so $\lambda x^K.M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(\omega^K) \rightsquigarrow \mathcal{I}(T) = \mathcal{I}(\omega^K \to T)$.

– Let $\dfrac{M_1 : \langle \Gamma_1 \vdash V \to T\rangle \quad M_2 : \langle \Gamma_2 \vdash V\rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1\ M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle}$ where $\Gamma_1 = (x_j^{L_j} : U_j)_n, (y_j^{K_j} : V_j)_m, \Gamma_2 = (x_j^{L_j} : U'_j)_n, (z_j^{S_j} : W_j)_p$ such that $\{y_1^{K_1}, \ldots, y_m^{K_m}\} \cap \{z_1^{S_1}, \ldots, z_p^{S_p}\} = \emptyset$ and $\Gamma_1 \sqcap \Gamma_2 = (x_j^{L_j} : U_j \sqcap U'_j)_n, (y_j^{K_j} : V_j)_m, (z_j^{S_j} : W_j)_p$.
Let $\forall\ 1 \le j \le n, P_j \in \mathcal{I}(U_j \sqcap U'_j)$, $\forall\ 1 \le j \le m, Q_j \in \mathcal{I}(V_j)$ and $\forall\ 1 \le j \le p, R_j \in \mathcal{I}(W_j)$. So, for all $j \in \{1, \ldots, n\}$, $P_j \in \mathcal{I}(U_j)$ and $P_j \in \mathcal{I}(U'_j)$. By hypothesis, $(M_1M_2)[(x_j^{L_j} := P_j)_n, (y_j^{K_j} := Q_j)_m, (z_j^{S_j} := R_j)_p] = AB \in \mathcal{M}$ where using lemma 4.2, $A = M_1[(x_j^{L_j} := P_j)_n, (y_j^{K_j} := Q_j)_m] \in \mathcal{M}$ and $B = M_2[(x_j^{L_j} := P_j)_n, (z_j^{S_j} := R_j)_p] \in \mathcal{M}$ and $A \diamond B$.
By IH, $A \in \mathcal{I}(V) \rightsquigarrow \mathcal{I}(T)$ and $B \in \mathcal{I}(V)$. Hence, $AB \in \mathcal{I}(T)$.

– Let $\dfrac{M : \langle (x_j^{L_j} : U_j)_n \vdash V_1\rangle \quad M : \langle (x_j^{L_j} : U_j)_n \vdash V_2\rangle}{M : \langle (x_j^{L_j} : U_j)_n \vdash V_1 \sqcap V_2\rangle}$. By IH, $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(V_1)$ and $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(V_2)$. Hence, $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(V_1 \sqcap V_2)$.

– Let $\dfrac{M : \langle (x_k^{L_k} : U_k)_n \vdash U\rangle}{M^{+j} : \langle (x_k^{j::L_k} : \overline{e}_j U_k)_n \vdash \overline{e}_j U\rangle}$ and $\forall\ 1 \le k \le n$, $N_k \in \mathcal{I}(\overline{e}_j U_k) = \mathcal{I}(U_k)^{+j}$. Then $\forall\ 1 \le k \le n$, $N_k = P_k^{+j}$ where $P_k \in \mathcal{I}(U_k)$. By lemma 10.1b, for all $k \in \{1, \ldots, n\}$, $P_k \in \mathcal{M}^{L_k}$. By the definition of the substitution, $\diamond\{M^{+j}\} \cup \{N_k \ / \ k \in \{1, \ldots, n\}\}$. By lemma 19.3, $\diamond\{M\} \cup \{P_k \ / \ k \in \{1, \ldots, n\}\}$. By lemma 18.5, $M[(x_k^{L_k} := P_k)_n] \in \mathcal{M}$. By IH, $M[(x_k^{L_k} := P_k)_n] \in \mathcal{I}(T)$. Hence, by lemma 19, $M^{+j}[(x_k^{j::L_k} := N_k)_n] = (M[(x_k^{L_k} := P_k)_n])^{+j} \in \mathcal{I}(U)^{+j} = \mathcal{I}(\overline{e}_j U)$.

– Let $\dfrac{M : \Phi \quad \Phi \sqsubseteq \Phi'}{M : \Phi'}$ where $\Phi' = \langle (x_j^{L_j} : U_j)_n \vdash U\rangle$. By lemma 3, we have $\Phi = \langle (x_j^{L_j} : U'_j)_n \vdash U'\rangle$, where for every $1 \le j \le n$, $U_j \sqsubseteq U'_j$ and $U' \sqsubseteq U$. By lemma 10.2, $N_j \in \mathcal{I}(U'_j)$, then, by IH, $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(U')$ and, by lemma 10.2, $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(U)$. □

Proof (Of lemma 13).

1. Let $y \in \mathcal{V}_2$ and $\mathcal{X} = \{M \in \mathcal{M}^{\oslash} \ / \ M \rhd_\beta^* x^{\oslash} N_1 \ldots N_k$ where $k \ge 0$ and $x \in \mathcal{V}_1$ or $M \rhd_\beta^* y^{\oslash}\}$. $\mathcal{X}$ is β-saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^{\oslash} \subseteq \mathcal{X} \subseteq \mathcal{M}^{\oslash}$. Take a β-interpretation $\mathcal{I}$ such that $\mathcal{I}(a) = \mathcal{X}$. If $M \in [Id_0]_\beta$, then M is closed and $M \in \mathcal{X} \rightsquigarrow \mathcal{X}$. Since $y^{\oslash} \in \mathcal{X}$ and $M \diamond y^{\oslash}$ then $My^{\oslash} \in \mathcal{X}$ and $My^{\oslash} \rhd_\beta^* x^{\oslash} N_1 \ldots N_k$ where $k \ge 0$ and $x \in \mathcal{V}_1$ or $My^{\oslash} \rhd_\beta^* y^{\oslash}$. Since M is closed and $x^{\oslash} \neq y^{\oslash}$, by lemma 1.2, $My^{\oslash} \rhd_\beta^* y^{\oslash}$. Hence, by lemma 20.4, $M \rhd_\beta^* \lambda y^{\oslash}.y^{\oslash}$ and, by lemma 1, $M \in \mathcal{M}^{\oslash}$.
Conversely, let $M \in \mathcal{M}^{\oslash}$ such that M is closed and $M \rhd_\beta^* \lambda y^{\oslash}.y^{\oslash}$. Let $\mathcal{I}$ be an β-interpretation and $N \in \mathcal{I}(a)$ such that $M \diamond N$. By lemma 10.1b, $N \in \mathcal{M}^{\oslash}$,

so $MN \in \mathcal{M}^{\oslash}$. Since $\mathcal{I}(a)$ is β-saturated and $MN \rhd_\beta^* N$, $MN \in \mathcal{I}(a)$ and hence $M \in \mathcal{I}(a) \rightsquigarrow \mathcal{I}(a)$. Hence, $M \in [Id_0]_\beta$.

2. By lemma 12 and lemma 9, $[Id'_1]_\beta = [\overline{e}_1 a \to \overline{e}_1 a]_\beta = [\overline{e}_1(a \to a)]_\beta = [Id_1] = [a \to a]_\beta^{+1} = [Id_0]_\beta^{+1}$. By 1., $[Id_0]_\beta^{+1} = \{M \in \mathcal{M}^{(1)} \ / \ M \rhd_\beta^* \lambda y^{(1)}.y^{(1)}\}$.
3. Let $y \in \mathcal{V}_2$, $\mathcal{X} = \{M \in \mathcal{M}^{\oslash} \ / \ M \rhd_\beta^* y^{\oslash}$ or $M \rhd_\beta^* x^{\oslash} N_1 ... N_k$ where $k \geq 0$ and $x \in \mathcal{V}_1\}$ and $\mathcal{Y} = \{M \in \mathcal{M}^{\oslash} \ / \ M \rhd_\beta^* y^{\oslash} y^{\oslash}$ or $M \rhd_\beta^* x^{\oslash} N_1 ... N_k$ or $M \rhd_\beta^* y^{\oslash}(x^{\oslash} N_1 ... N_k)$ where $k \geq 0$ and $x \in \mathcal{V}_1\}$. $\mathcal{X}$, $\mathcal{Y}$ are β-saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^{\oslash} \subseteq \mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}^{\oslash}$. Let $\mathcal{I}$ be a β-interpretation such that $\mathcal{I}(a) = \mathcal{X}$ and $\mathcal{I}(b) = \mathcal{Y}$. If $M \in [D]_\beta$, then M is closed (hence $M \diamond y^{\oslash}$) and $M \in (\mathcal{X} \cap (\mathcal{X} \rightsquigarrow \mathcal{Y})) \rightsquigarrow \mathcal{Y}$. Since $y^{\oslash} \in \mathcal{X}$ and $y^{\oslash} \in \mathcal{X} \rightsquigarrow \mathcal{Y}$, $y^{\oslash} \in \mathcal{X} \cap (\mathcal{X} \rightsquigarrow \mathcal{Y})$ and $My^{\oslash} \in \mathcal{Y}$. Since $x^{\oslash} \neq y^{\oslash}$, by lemma 1.2, $My^{\oslash} \rhd_\beta^* y^{\oslash} y^{\oslash}$. Hence, by lemma 20.4, $M \rhd_\beta^* \lambda y^{\oslash}.y^{\oslash} y^{\oslash}$ and, by lemma 1, $\mathrm{d}(M) = \oslash$ and $M \in \mathcal{M}^{\oslash}$.
   Conversely, let $M \in \mathcal{M}^{\oslash}$ such that M is closed and $M \rhd_\beta^* \lambda y^{\oslash}.y^{\oslash} y^{\oslash}$. Let $\mathcal{I}$ be an β-interpretation and $N \in \mathcal{I}(a \sqcap (a \to b)) = \mathcal{I}(a) \cap (\mathcal{I}(a) \rightsquigarrow \mathcal{I}(b))$ such that $M \diamond N$. By lemma 10.1b and lemma 18.1, $N \in \mathcal{M}^{\oslash}$ and $N \diamond N$. So $NN, MN \in \mathcal{M}^{\oslash}$. Since $\mathcal{I}(b)$ is β-saturated, $NN \in \mathcal{I}(b)$ and $MN \rhd_\beta^* NN$, we have $MN \in \mathcal{I}(b)$ and hence $M \in \mathcal{I}(a \sqcap (a \to b)) \rightsquigarrow \mathcal{I}(b)$. Therefore, $M \in [D]_\beta$.
4. Let $f, y \in \mathcal{V}_2$ and take $\mathcal{X} = \{M \in \mathcal{M}^{\oslash} \ / \ M \rhd_\beta^* (f^{\oslash})^n (x^{\oslash} N_1 ... N_k)$ or $M \rhd_\beta^* (f^{\oslash})^n y^{\oslash}$ where $k, n \geq 0$ and $x \in \mathcal{V}_1\}$. $\mathcal{X}$ is β-saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^{\oslash} \subseteq \mathcal{X} \subseteq \mathcal{M}^{\oslash}$. Let $\mathcal{I}$ be a β-interpretation such that $\mathcal{I}(a) = \mathcal{X}$. If $M \in [Nat_0]_\beta$, then M is closed and $M \in (\mathcal{X} \rightsquigarrow \mathcal{X}) \rightsquigarrow (\mathcal{X} \rightsquigarrow \mathcal{X})$. We have $f^{\oslash} \in \mathcal{X} \rightsquigarrow \mathcal{X}$, $y^{\oslash} \in \mathcal{X}$ and $\diamond\{M, f^{\oslash}, y^{\oslash}\}$ then $Mf^{\oslash} y^{\oslash} \in \mathcal{X}$ and $Mf^{\oslash} y^{\oslash} \rhd_\beta^* (f^{\oslash})^n (x^{\oslash} N_1 ... N_k)$ or $Mf^{\oslash} y^{\oslash} \rhd_\beta^* (f^{\oslash})^n y^{\oslash}$ where $n \geq 0$ and $x \in \mathcal{V}_1$. Since M is closed and $\{x^{\oslash}\} \cap \{y^{\oslash}, f^{\oslash}\} = \emptyset$, by lemma 1.2, $Mf^{\oslash} y^{\oslash} \rhd_\beta^* (f^{\oslash})^n y^{\oslash}$ where $n \geq 1$. Hence, by lemma 20.4, $M \rhd_\beta^* \lambda f^{\oslash}.f^{\oslash}$ or $M \rhd_\beta^* \lambda f^{\oslash}.\lambda y^{\oslash}.(f^{\oslash})^n y^{\oslash}$ where $n \geq 1$. Moreover, by lemma 1, $\mathrm{d}(M) = \oslash$ and $M \in \mathcal{M}^{\oslash}$.
   Conversely, let $M \in \mathcal{M}^{\oslash}$ such that M is closed and $M \rhd_\beta^* \lambda f^{\oslash}.f^{\oslash}$ or $M \rhd_\beta^* \lambda f^{\oslash}.\lambda y^{\oslash}.(f^{\oslash})^n y^{\oslash}$ where $n \geq 1$. Let $\mathcal{I}$ be an β-interpretation, $N \in \mathcal{I}(a \to a) = \mathcal{I}(a) \rightsquigarrow \mathcal{I}(a)$ and $N' \in \mathcal{I}(a)$ such that $\diamond\{M, N, N'\}$. By lemma 10.1b, $N, N' \in \mathcal{M}^{\oslash}$, so $MNN', (N)^m N' \in \mathcal{M}^{\oslash}$, where $m \geq 0$. We show, by induction on $m \geq 0$, that $(N)^m N' \in \mathcal{I}(a)$. Since $MNN' \rhd_\beta^* (N)^m N'$ where $m \geq 0$ and $(N)^m N' \in \mathcal{I}(a)$ which is β-saturated, then $MNN' \in \mathcal{I}(a)$. Hence, $M \in (\mathcal{I}(a) \rightsquigarrow \mathcal{I}(a)) \to (\mathcal{I}(a) \rightsquigarrow \mathcal{I}(a))$ and $M \in [Nat_0]_\beta$.
5. By lemma 12, $[Nat_1] = [\overline{e}_1 Nat_0] = [Nat_0]^{+1}$. By 4., $[Nat_1] = [Nat_0]^{+1} = \{M \in \mathcal{M}^{(1)} \ / \ M \rhd_\beta^* \lambda f^{(1)}.f^{(1)}$ or $M \rhd_\beta^* \lambda f^{(1)}.\lambda y^{(1)}.(f^{(1)})^n y^{(1)}$ where $n \geq 1\}$.
6. Let $f, y \in \mathcal{V}_2$ and take $\mathcal{X} = \{M \in \mathcal{M}^{\oslash} \ / \ M \rhd_\beta^* x^{\oslash} P_1 ... P_l$ or $M \rhd_\beta^* f^{\oslash}(x^{\oslash} Q_1 ... Q_n)$ or $M \rhd_\beta^* y^{\oslash}$ or $M \rhd_\beta^* f^{\oslash} y^{(1)}$ where $l, n \geq 0$ and $\mathrm{d}(Q_i) \succeq (1)\}$. $\mathcal{X}$ is β-saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^{\oslash} \subseteq \mathcal{X} \subseteq \mathcal{M}^{\oslash}$. Let $\mathcal{I}$ be a β-interpretation such that $\mathcal{I}(a) = \mathcal{X}$. If $M \in [Nat'_0]_\beta$, then M is closed and $M \in (\mathcal{X}^{+1} \rightsquigarrow \mathcal{X}) \rightsquigarrow (\mathcal{X}^{+1} \rightsquigarrow \mathcal{X})$. Let $N \in \mathcal{X}^{+1}$ such that $N \diamond f^{\oslash}$. We have $N \rhd_\beta^* x^{(1)} P_1^{+1} ... P_k^{+1}$ or $N \rhd_\beta^* y^{(1)}$, then $f^{\oslash} N \rhd_\beta^* f^{\oslash}(x^{(1)} P_1^{+1} ... P_k^{+1}) \in \mathcal{X}$ or $N \rhd_\beta^* f^{\oslash} y^{(1)} \in \mathcal{X}$, thus $f^{\oslash} \in \mathcal{X}^{+1} \rightsquigarrow \mathcal{X}$. We have $f^{\oslash} \in \mathcal{X}^{+1} \rightsquigarrow \mathcal{X}$,

$y^{(1)} \in \mathcal{X}^{+1}$ and $\diamond\{M, f^{\oslash}, y^{(1)}\}$, then $Mf^{\oslash}y^{(1)} \in \mathcal{X}$. Since M is closed and $\{x^{\oslash}, x^{(1)}\} \cap \{y^{(1)}, f^{\oslash}\} = \emptyset$, by lemma 1.2, $Mf^{\oslash}y^{(1)} \rhd_{\beta}^{*} f^{\oslash}y^{(1)}$. Hence, by lemma 20.4, $M \rhd_{\beta}^{*} \lambda f^{\oslash}.f^{\oslash}$ or $M \rhd_{\beta}^{*} \lambda f^{\oslash}.\lambda y^{(1)}.f^{\oslash}y^{(1)}$. Moreover, by lemma 1, $\mathrm{d}(M) = \oslash$ and $M \in \mathcal{M}^{\oslash}$.
Conversely, let $M \in \mathcal{M}^{\oslash}$ such M is closed and $M \rhd_{\beta}^{*} \lambda f^{\oslash}.f^{\oslash}$ or $M \rhd_{\beta}^{*} \lambda f^{\oslash}.\lambda y^{(1)}.f^{\oslash}y^{(1)}$. Let $\mathcal{I}$ be an β-interpretation, $N \in \mathcal{I}(\overline{e}_1 a \to a) = \mathcal{I}(a)^{+1} \rightsquigarrow \mathcal{I}(a)$ and $N' \in \mathcal{I}(a)^{+1}$ where $\diamond\{M, N, N'\}$. By lemma 10.1b, $N \in \mathcal{M}^{\oslash}$ and $N' \in \mathcal{M}^{(1)}$, so $MNN', NN' \in \mathcal{M}^{\oslash}$. Since $MNN' \rhd_{\beta}^{*} NN'$, $NN' \in \mathcal{I}(a)$ and $\mathcal{I}(a)$ is β-saturated, then $MNN' \in \mathcal{I}(a)$. Hence, $M \in (\mathcal{I}(a)^{+1} \rightsquigarrow \mathcal{I}(a)) \to (\mathcal{I}(a)^{+1} \rightsquigarrow \mathcal{I}(a))$ and $M \in [Nat'_0]$. $\square$