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First-order Temporal Logic with Fixpoint Operators over the Natural Numbers

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FIRST-ORDER TEMPORAL LOGIC WITH FIXPOINT OPERATORS OVER THE NATURAL NUMBERS

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Abstract. We show that the satisfiability problem for monodic first-order linear time temporal logic with temporal fixpoint operators and no equality over the flow of the natural numbers is in EXPSPACE (2EXPTIME), if the purely first-order part of the language is restricted to a fragment of first-order logic that lies in EXPSPACE (2EXPTIME). The same upper complexity bound holds for the finite satisfiability problem. For the monodic packed fragment with equality we show 2EXPTIME-completeness and EXPSPACE-completeness for the bounded-variable or bounded-arity case. We also consider the monodic guarded and loosely guarded fragments with temporal fixpoints as well as domain-side least fixpoints and show 2EXPTIME-completeness (EXPSPACE-completeness for the bounded-variable or bounded-arity case) of the satisfiability problem over arbitrary domains.

Contents

1. Introduction

Pnueli was the first who proposed the use of temporal logic as a means of specifying and verifying correctness properties of programs [46]. For plain deterministic sequential programs, the correctness of which is essentially adherence to an input/output specification, a formalism such a Hoare's logic can also be used [26]. Temporal logic is particularly suited for nonterminating, reactive systems such as operating systems and network communication protocols. Since the work of Pnueli, both linear-time and branching-time temporal logics have found wide applications in program specification and verification. An important distinction is between deductive verification, in which proof-theoretic methods are used and which is not fully automated, and model checking, which is a technique for verifying finite state concurrent systems in a completely mechanized way. Today, model checking is of great commercial importance for the verification of hardware designs, network protocols, etc. Let us note two important works in the 1980s, one of Clarke, Emerson, Sistla [11] and one of Clarke, Emerson [10], which proposed model checking algorithms for verifying branching-time temporal-logic specifications of concurrent systems. Another notable application of temporal logic is in the field of temporal databases, where relational data are enriched with a temporal dimension. See chapters 14,15, and 16 of [19] for a discussion of relevant topics. Temporal logics also have a role to play in various subareas of artificial intelligence such as knowledge representation for application domains that require time to be taken into consideration, in multi-agent systems, where knowledge and belief of agents may be time-evolving, etc.

A general observation about the use of logics in computer science is that a compromise is generally sought between the expressiveness of a logical system and its computational difficulty. For a given application, a logic has to be designed so that it is sufficiently expressive for its intended purpose and the associated problems (such as validity checking, satisfiability checking, model checking) are reasonably tractable. Consider first-order logic, which is generally considered as a very expressive formalism for talking about relational structures: its applicability is severely limited due to the fact that it is undecidable. Many fragments of first-order logic, however, are expressive enough for some applications and their complexity allows practical algorithms. Along similar lines, it is interesting to investigate whether it is possible to take a logical system of known complexity, enhance its expressiveness while still remaining in the same complexity class. Our work here is in this direction. Starting from 'monodic' fragments of first-order temporal logic, we add fixpoint operators to the temporal side, thus adding second-order expressiveness, without increasing the complexity.

1.1. **Relevant work.** In what follows, we will restrict our attention to linear-time temporal logics. We will discuss briefly some temporal logics, starting with the simple propositional temporal logic (PTL) and continuing with more expressive logics, so that it is made more clear how our results relate to existing work.

It has been proved that for the time flow of the natural numbers, which is the most commonly used flow for applications in specification and verification, PTL is expressively complete for the monadic first-order theory of linear order over the naturals [35, 17], which is known to be non-elementary [52]. This "expressive completeness" result, however, does not mean that there are not circumstances in which a more expressive formalism is required.

The observation that we cannot express in propositional temporal logic properties that can be expressed by regular grammars led Pierre Wolper to consider an extended temporal logic, called ETL, which contains temporal operators that correspond to regular expressions [62]. Even though ETL is more expressive than PTL, its satisfiability problem lies in the same complexity class, i.e. it is complete for PSPACE [63, 49]. An obvious drawback of this approach is that we have to consider an infinity of grammar operators in order to get the full expressive power of ETL.

Similar in spirit are proposals for using ω-regular automata as temporal operators. Vardi and Wolper consider in [61] the use of finite automata over ω-words with various acceptance conditions (finite, looping, and Büchi) as well as alternating automata. It turns out that in all cases the logics are equally expressive and that the corresponding satisfiability problems are all in

PSPACE. This is particularly interesting for the case of alternating automata operators, since alternating automata are exponentially more succinct than nondeterministic automata over infinite words. In [38], two-way alternating automata, which can be transformed to language-equivalent nondeterministic automata with an exponential blow-up, are considered as temporal connectives. The satisfiability problem is shown to be in PSPACE.

Let us note at this point that ω-automata are as expressive as the monadic second-order theory of one successor function (S1S) [7], which is non-elementary [43]. Another extension of PTL, quantified PTL or QPTL [50], in which quantification over propositional atoms is allowed, is equally expressive as S1S and its satisfiability problem is non-elementary [51].

Extensions with fixpoint operators have also been considered. They are similar to the fixpoint extensions of propositional dynamic logic presented in [47, 36, 37]. Banieqbal and Barringer introduce νTL, the propositional temporal logic with least (μ) and greatest (ν) fixpoint operators and the tomorrow operator, which is expressively equivalent to ETL [4]. They give a decision procedure for the satisfiability problem that needs exponential time and space. Vardi extends νTL with the yesterday operator and gives an automata-theoretic decision procedure for satisfiability that needs polynomial space [58].

In propositional temporal logic, we define semantics by attaching to each moment the set of propositional atoms that are true at that particular moment. We can greatly increase the expressive power by attaching a first-order structure to each moment and using a first-order temporal language to talk about these much more complex temporal structures. Unfortunately, full first-order temporal logic (FOTL) turns out to be highly undecidable. Unpublished results of Per Lindström and Dana Scott first established that the first-order temporal logics with temporal operators 'sometime in the future' (F) and 'sometime in the past' (P) over the naturals, the integers, and the reals are not even recursively enumerable. There is a general result due to Mark Reynolds that shows several first-order temporal logics to be not recursively axiomatizable by encoding first-order arithmetic in them (Theorem 4.6.1 in [18]). Let us note, however, that there are axiomatizations for the first-order temporal logics of F and P over the class of all linear flows and over the flow of the rationals.

After a series of negative results (see, for example, [1, 42, 54, 55]), Hodkinson, Wolter, and Zakharyaschev presented expressive 'monodic' fragments of FOTL that are decidable over various classes of flows of time [32]. The monodicity requirement essentially limits the interaction between the temporal and the first-order dimension, by imposing the restriction that subformulas beginning with a temporal topmost operator have at most one free variable. In order to get decidability, the purely first-order part of the language will also have to be restricted to a decidable fragment.

Lower and upper complexity bounds have been proved for monodic first-order temporal logics over the flow of the naturals [30]. An EXPSPACE-hardness result can be shown by encoding the 2^n-corridor tiling problem using just formulas that are in the one-variable fragment and contain only the 'always in the future' (G) temporal operator. An EXPSPACE (2EXPTIME) upper bound can also be obtained, when the first-order part lies in EXPSPACE (2EXPTIME). The applicability of this result is immediately obvious if we notice that various decidable classes of first-order logic, such as the monadic class (with unary function symbols or without), the Bernays-Schönfinkel class (formulas with a $\exists^\star\forall^\star$ quantifier prefix), the Gödel-Kalmár-Schütte class (formulas with a $\exists^\star\forall^2\exists^\star$ quantifier prefix), the Ackermann class (formulas with a $\exists^\star\forall\exists^\star$ quantifier prefix), and the two-variable class (formulas with two distinct individual variables) are all complete for NEXPTIME and hence in EXPSPACE [6]. Thus, we deduce that the satisfiability problems for the corresponding fragments of monodic FOTL over the naturals are complete for EXPSPACE.

Wolter and Zakharyaschev show in [64] that the monodic fragment over the naturals has a finite Hilbert-style axiomatization, but becomes not recursively enumerable when equality is added to the language. Degtyarev, Fisher, and Lisitsa prove that even the simpler monodic two-variable fragment with equality over the naturals is not recursively enumerable [12]. Hodkinson, however, has obtained a positive result by showing that if we restrict the first-order part to the packed fragment with equality, we get decidable monodic fragments over various classes of flows of time [28]. Complexity results for such monodic guarded fragments with equality are given in [29].

The addition of function symbols has been considered by Hussak in [34]. He extends monadic monodic fragments with flexible functions and without equality and shows decidability for some flows of time and EXPSPACE-completeness for the flow of the naturals. The addition of one rigid function makes the logic not recursively enumerable. However, further restrictions can be applied on the way terms interact with quantifiers and regain decidability for these 'term monodic' fragments.

In an effort to get even more powerful logics, it seems very interesting to examine whether we can extend decidable monodic fragments of FOTL with fixpoint operators and still get decidable logics. Indeed, it has been shown in [13] that this is possible. The satisfiability problem for monodic FOTL with temporal fixpoint operators is decidable over various classes of flows of time, given that the corresponding first-order part is decidable. The complexity, however, of these decidable fragments was left open.

1.2. **Contribution.** In this report, we consider monodic FOTL with fixpoint operators over the flow of the natural numbers. The language contains the usual Boolean connectives, first-order quantifiers, the temporal operators $\circ$ ('tomorrow'), $\bullet$ ('yesterday'), $\ominus$ ('yesterday', if yesterday exists'), as well as least (μ) and greatest (ν) temporal fixpoint operators. These operators express easily Kamp's temporal connectives $\mathscr{U}$ ('until') and $\mathscr{S}$ ('since')

$$\phi \mathscr{U} \psi \equiv \mu X[\circ\psi \vee \circ(\phi \wedge X)] \qquad\qquad \phi \mathscr{S} \psi \equiv \mu X[\bullet\psi \vee \bullet(\phi \wedge X)] \ .$$

The monodicity restriction, which is essential for decidability, can be briefly stated as 'every subformula that begins with a temporal or fixpoint operator has at most one free individual variable and every subformula that begins with a quantifier has no free fixpoint variables'. Even with this restriction we get the expressiveness of full first-order logic, full temporal logic with fixpoint operators, and even more.

We present new results regarding the computational complexity of several such monodic logics over the naturals. First, we show the following general theorems.

- If the first-order part is in EXPSPACE (2EXPTIME), then the satisfiability problem for the logic is also in EXPSPACE (2EXPTIME).
- The same holds for the finite satisfiability problem, i.e. the problem of deciding whether a formula has a temporal model of finite domain.
- As corollaries, we obtain, for example, that the monadic, the one-variable, and the two-variable monodic first-order temporal logics with temporal fixpoint operators over the naturals are complete for EXPSPACE. EXPSPACE-hardness follows from Theorem 3.1 of [30].

The interesting observation is that, similarly to the propositional case, enhancing the expressiveness on the temporal side with fixpoint operators does not result in an increase of the complexity. We also consider the case of the monodic packed fragment with equality and prove the following.

- The (finite) satisfiability problem for the monodic packed fragment with equality and temporal fixpoint operators over the naturals is complete for 2EXPTIME.
- In the bounded-variable or in the bounded-arity case the (finite) satisfiability problem for this fragment is complete for EXPSPACE.

We investigate the extension of the monodic packed fragment with least fixpoints on the domain side, thus having both time-fixpoints and domain-fixpoints, and obtain the following.

- The satisfiability problem for the monodic (loosely) guarded fragment with equality, least and greatest temporal fixpoints and least domain-fixpoints over the naturals is complete for 2EXPTIME.
- Again, if we bound the variables or the arity of the symbols, we get EXPSPACE-completeness.

In order to prove these results, we extend the 'quasimodel' technique of [32], which is reminiscent of the well-known filtration technique for modal logics, and the 'well-founded premodel' technique of [53, 58]. Quasimodels are objects that encode models. Instead of attaching a first-order structure to each moment, we can say roughly that a quasimodel records at each moment the subformulas that hold at each domain point. The monodicity restriction allows special functions called runs to encode temporal consistency. The notion of well-foundedness is useful for characterizing the way fixpoints are evaluated: a least fixpoint involves finite looping, whereas a greatest fixpoint may involve infinite looping. The crucial proposition is that a sentence has a temporal model if and only if it has a well-founded quasimodel. The complexity results are obtained by extending the automata-theoretic techniques for PTL [58, 59]. That is, (finite) satisfiability of a sentence is reduced to the nonemptiness problem of a large and complex Büchi automaton. This approach results in a clean proof that separates well the logical from the combinatorial part.

We believe that apart from the importance of our results in themselves, there are many interesting theoretical applications. The logics we consider are very powerful and they subsume expressively a very wide range of other logics. Thus, we provide a general framework for encoding logics that have a temporal dimension such as various temporal description logics and logics of knowledge and belief. This means that complexity results can be obtained for many logics as immediate corollaries of the theorems presented here.

1.3. **Layout of report.** In Section 2 we present the syntax of first-order temporal logic with fixpoint operators. We continue in Section 3 with the definition of semantics, which is shown to be well-defined. Section 4 introduces the notions of fixpoint approximants and signatures. With the help of signatures we will formalize later the idea that least fixpoints need a finite number of steps to be evaluated, i.e. the idea of well-founded regeneration of least fixpoint formulas. The monodic fragment is defined in Section 5. The quasimodel technique of [32] is described and extended with the notion of well-foundedness of [53] in Section 6. We conclude Section 6 with the crucial theorem that a sentence of our language has a (finite) model if and only if it is satisfied in a (finitary) well-founded quasimodel. In Section 7 we show upper complexity bounds. Given a sentence ϕ, whose (finite) satisfiability we want to check, we construct an automaton that accepts the (finitary) quasimodels — stripped down of their runs — that satisfy ϕ. We discuss in Section 8 that the technique of [28] for the monodic packed fragment directly extends to our case, when fixpoint operators are added. Domain-side least fixpoint operators are considered in Section 9 as an extension of the monodic packed fragment. Section 10 is devoted to presenting how description logics are extended with a temporal dimension and showing how the more general reasoning tasks of a wide range of such logics can be immediately reduced to the satisfiability problem of monodic first-order temporal logic with fixpoint operators. For many cases tight complexity bounds can be obtained as corollaries. The report is concluded in Section 11 with a summary of the results presented here and a brief discussion about open questions that relate to our work.

2. Syntax

We present formally the syntax of full first-order temporal logic with temporal fixpoint operators. First, we enumerate the various symbols that make up the alphabet and proceed to define terms and formulas. Some useful definitions for individual and fixpoint variables are also introduced.

We fix a *first-order signature*, which is a tuple $\sigma = (\mathcal{P}, \mathcal{P}_0, \mathcal{F}, \mathcal{C}, \mathsf{ar})$. $\mathcal{P} = \{P_0, P_1, \ldots\}$ is the set of *predicate symbols*, $\mathcal{P}_0 = \{p_0, p_1, \ldots\}$ the set of *propositional variables*, $\mathcal{F} = \{f_0, f_1, \ldots\}$ the set of *function symbols*, and $\mathcal{C} = \{c_0, c_1, \ldots\}$ the set of *individual constants.* To each predicate symbol and each function symbol we associate a positive integer called its *arity* through the function $\mathsf{ar} : \mathcal{P} \cup \mathcal{F} \to \mathbb{N} \setminus \{0\}$. We also fix a countably infinite set $\mathcal{V} = \{x_0, x_1, \ldots, y, z, \ldots\}$ of *individual variables* and a countably infinite set $\mathcal{X} = \{X_0, X_1, \ldots, Y, Z, \ldots\}$ of *fixpoint variables.* The alphabet will also include the Boolean connectives $\neg, \wedge, \vee$, the universal quantifier $\forall$, the existential quantifier $\exists$, the temporal operators $\bigcirc$ (tomorrow), $\bullet$ (yesterday), $\ominus$ (yesterday, if yesterday exists), and the fixpoint operators μ (least fixpoint) and ν (greatest fixpoint). We also include the punctuation symbols left parenthesis, right parenthesis, and comma. So, let the alphabet be

$$\Sigma = \mathcal{P} \cup \mathcal{P}_0 \cup \mathcal{F} \cup \mathcal{C} \cup \mathcal{V} \cup \mathcal{X} \cup \{\neg, \wedge, \vee, \forall, \exists, \bigcirc, \bullet, \ominus, \mu, \nu, (\, , \,), \, , \} \; .$$

See Table 1 for a listing of the various symbols that make up the alphabet. The Boolean operators $\to, \leftrightarrow$ will be treated as abbreviations.

Definition 1 (terms)**.** The *terms* are constructed using only symbols from $\mathcal{F}, \mathcal{C}, \mathcal{V}$ and punctuation symbols. Let us call the set that contains all these symbols $\Sigma_{\mathcal{T}}$. Clearly, $\Sigma_{\mathcal{T}} \subseteq \Sigma$. The set of terms $\mathcal{T}$ is the smallest subset of $\Sigma_{\mathcal{T}}^\star$ (Kleene closure of $\Sigma_{\mathcal{T}}$) that satisfies the following conditions.

- $\mathcal{T}$ includes $\mathcal{V}$, i.e. $\mathcal{V} \subseteq \mathcal{T}$.
- $\mathcal{T}$ includes $\mathcal{C}$, i.e. $\mathcal{C} \subseteq \mathcal{T}$.
- For any function symbol f, if $t_1, \ldots, t_m$ are in $\mathcal{T}$ $(m = \mathsf{ar}[f])$, then $f(t_1, \ldots, t_m) \in \mathcal{T}$.

Definition 2 ($\mathcal{FOTL}_{\mu\nu}$)**.** The set of *first-order temporal formulas with fixpoint operators* $\mathcal{FOTL}_{\mu\nu}$ is the smallest subset of $\Sigma^\star$ that satisfies the following conditions.

- $\mathcal{FOTL}_{\mu\nu}$ includes $\mathcal{P}_0$, i.e. $\mathcal{P}_0 \subseteq \mathcal{FOTL}_{\mu\nu}$.
- $\mathcal{FOTL}_{\mu\nu}$ includes $\mathcal{X}$, i.e. $\mathcal{X} \subseteq \mathcal{FOTL}_{\mu\nu}$.
- For any predicate symbol P, if $t_1, \ldots, t_n \in \mathcal{T}$ $(n = \mathsf{ar}[P])$, then $P(t_1, \ldots, t_n) \in \mathcal{FOTL}_{\mu\nu}$.
- For any unary operator $\circ \in \{\neg, \bigcirc, \bullet, \ominus\}$, if $\phi \in \mathcal{FOTL}_{\mu\nu}$, then $\circ\phi \in \mathcal{FOTL}_{\mu\nu}$.
- For any binary operator $\otimes \in \{\wedge, \vee\}$, if $\phi_1, \phi_2 \in \mathcal{FOTL}_{\mu\nu}$, then $(\phi_1 \otimes \phi_2) \in \mathcal{FOTL}_{\mu\nu}$.
- For any quantifier $Q \in \{\forall, \exists\}$, if $x \in \mathcal{V}$ and $\phi \in \mathcal{FOTL}_{\mu\nu}$, then $Qx\phi \in \mathcal{FOTL}_{\mu\nu}$.
- For any fixpoint operator $f \in \{\mu, \nu\}$, if X is a fixpoint variable, $\phi \in \mathcal{FOTL}_{\mu\nu}$, and all free occurrences of X in ϕ are positive, then $fX\phi \in \mathcal{FOTL}_{\mu\nu}$.

We will write $\mathcal{FOTL}_{\mu\nu}[\sigma]$ when we want to make explicit that the signature of the language is σ.

In a (sub)formula $Qx\phi$, where Q is a quantifer, we say that Q *binds* x and that the occurrences of x in ϕ are *bound.* If an occurrence of an individual variable y in ϕ is not bound by any quantifier, then we say that it is *free.* For example, the first occurrence of y in

Table 1. The alphabet of first-order temporal logic with fixpoint operators.

predicate symbols	$\mathcal{P}$	$P_0, P_1, \ldots$	boolean connectives	$\neg, \wedge, \vee$
propositional variables	$\mathcal{P}_0$	$p_0, p_1, \ldots$	quantifiers	$\forall, \exists$
function symbols	$\mathcal{F}$	$f_0, f_1, \ldots$	temporal operators	$\bigcirc, \bullet, \ominus$
individual constants	$\mathcal{C}$	$c_0, c_1, \ldots$	fixpoint operators	μ, ν
individual variables	$\mathcal{V}$	$x_0, x_1, \ldots$	parentheses & comma	( ) ,

$\phi = \forall x[Rxy \wedge \exists y \forall z Pxyz]$ is free, but the second one is bound. Similarly, we define bound and free fixpoint variables.

We say that a free occurrence of a fixpoint variable is *positive* if it falls under an even number of negations and *negative* if it falls under an odd number of negations. A $\mathcal{FOTL}_{\mu\nu}$-formula is a *fp-sentence* if it has no free fixpoint variables. It is a *dom-sentence* if it has no free individual variables. It is a *sentence* if it is both a fp-sentence and a dom-sentence.

We will often abbreviate consecutive occurrences of temporal operators with a power-like notation. That is, $\bigcirc^3$ means $\bigcirc\bigcirc\bigcirc$. The usual parenthesis elimination conventions, such as omitting the outermost parentheses, etc., will be used subsequently whenever convenient. We denote the set of subformulas of ϕ by $\mathsf{sbf}[\phi]$.

Definition 3 (μ-vars, ν-vars, $\mu\nu$-vars, fp-free, fp-vars, vars, fvars, bvars)**.** Let ϕ be a $\mathcal{FOTL}_{\mu\nu}$-formula. We define μ-$\mathsf{vars}[\phi]$ to be the set of fixpoint variables bound by least fixpoint operators in ϕ, ν-$\mathsf{vars}[\phi]$ the set of fixpoint variables bound by greatest fixpoint operators in ϕ, $\mu\nu$-$\mathsf{vars}[\phi] = \mu$-$\mathsf{vars}[\phi] \cup \nu$-$\mathsf{vars}[\phi]$, fp-$\mathsf{free}[\phi]$ the set of free fixpoint variables in ϕ, and fp-$\mathsf{vars}[\phi]$ the set of all fixpoint variables that appear in ϕ. Obviously, it holds that

$$\mathsf{fp}\text{-}\mathsf{vars}[\phi] = \mu\nu\text{-}\mathsf{vars}[\phi] \cup \mathsf{fp}\text{-}\mathsf{free}[\phi] \ .$$

We denote by $\mathsf{vars}[\phi]$ the set of individual variables that appear in ϕ. $\mathsf{fvars}[\phi]$ is the set of individual variables that are free in ϕ, and $\mathsf{bvars}[\phi]$ the set of those that are bound by quantifiers. Clearly, $\mathsf{vars}[\phi] = \mathsf{fvars}[\phi] \cup \mathsf{bvars}[\phi]$.

Note that $\mathsf{fvars}[\phi]$ and $\mathsf{bvars}[\phi]$ are not necessarily disjoint, as the formula $(Px \wedge \exists y \forall x Rxy)$ illustrates. The same holds for μ-$\mathsf{vars}[\phi]$ and ν-$\mathsf{vars}[\phi]$ as well as for $\mu\nu$-$\mathsf{vars}[\phi]$ and fp-$\mathsf{free}[\phi]$. An example for the former would be the formula

$$\phi = \mu Z(\exists x Px \vee \bullet^3 Z) \wedge \nu Z(\exists x \neg Rxx \wedge \ominus Z) \ ,$$

for which μ-$\mathsf{vars}[\phi] = \nu$-$\mathsf{vars}[\phi] = \{Z\}$, and for the latter the formula

$$\psi = Y \wedge \mu Y(\exists x Px \vee \bigcirc X \vee \bullet Y) \ ,$$

for which $\mu\nu$-$\mathsf{vars}[\psi] = \{Y\}$ and fp-$\mathsf{free}[\psi] = \{X, Y\}$.

3. Semantics

The move from propositional temporal logic to first-order temporal logic opens a whole range of options as to how semantics is defined. Do we have the same domain of individuals at every moment? Are individual variable assignments time-dependent? See Chapter 4 of [18] for a discussion. Garson gives a very interesting survey in [20] about the available choices when formalizing quantified modal logics. We have opted here for constant domains, rigid constants and functions, and flexible predicates. After defining formally semantics, we show that it is indeed well-defined (this is not immediately obvious because of the temporal fixpoint operators). We also define the notions of equivalence, satisfiability, equisatisfiability, substitution and proceed to show a 'fixpoint unfolding' lemma, which reflects syntactically the definition of fixpoints.

In order to define semantics for $\mathcal{FOTL}_{\mu\nu}$-formulas, we need to specify a *first-order temporal structure* $\mathfrak{M} = \big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big)$ over the signature σ. The *underlying frame* is the flow of the natural numbers under their usual ordering. $\mathcal{D}$ is a non-empty set called the *domain* of $\mathfrak{M}$. Sometimes we will write $\mathsf{dom}(\mathfrak{M})$ to refer to the domain of $\mathfrak{M}$. I is a function that maps each moment $u \in \mathbb{N}$ to a first-order structure I_u over the signature σ.

$$I_u = (\mathcal{D}, \cdot^{I_u}) \qquad P^{I_u} \subseteq \mathcal{D}^n,\ n = \mathsf{ar}[P] \qquad p^{I_u} \in \{0,1\} \qquad f^{I_u} : \mathcal{D}^m \to \mathcal{D},\ m = \mathsf{ar}[f] \qquad c^{I_u} \in \mathcal{D}$$

$\cdot^{I_w}$ maps each predicate symbol $P \in \mathcal{P}$ to a n-ary ($n = \mathsf{ar}[P]$) relation on $\mathcal{D}$, each propositional variable $p \in \mathcal{P}_0$ to an element of $\{0,1\}$ (1 represents truth, and 0 falsity), each function symbol $f \in \mathcal{F}$ to a function from $\mathcal{D}^m$ ($m = \mathsf{ar}[f]$) to $\mathcal{D}$ and each individual constant $c \in \mathcal{C}$ to an element of the domain. We make the assumption of *rigid constants and functions*, i.e. for any $c \in \mathcal{C}$, any $f \in \mathcal{F}$, and any $u, v \in \mathbb{N}$, $c^{I_u} = c^{I_v}$ and $f^{I_u} = f^{I_v}$. In order to valuate terms we also need an *individual variable assignment*, $h : \mathcal{V} \to \mathcal{D}$, which maps each individual variable to an element of the domain. Hence, the *valuation of terms* is defined as follows for $x \in \mathcal{V}$, $c \in \mathcal{C}$, $f \in \mathcal{F}$, $m = \mathsf{ar}[f]$.

$$[\![x]\!]_h^{I_u} = h(x) \qquad [\![c]\!]_h^{I_u} = c^{I_u} \qquad [\![f(t_1,\ldots,t_m)]\!]_h^{I_u} = f^{I_u}([\![t_1]\!]_h^{I_u},\ldots,[\![t_m]\!]_h^{I_u})$$

We also fix a *fixpoint variable assignment* $g : \mathcal{X} \to \wp(\mathbb{N})$. Intuitively, g maps each fixpoint variable to the set of moments in which it is true. Now, we can define the truth relation inductively as follows.

$$\begin{aligned}
\big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models p &\iff p^{I_u} = 1 \\
\big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models X &\iff u \in g(X) \\
\big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models P(t_1,\ldots,t_n) &\iff \big([\![t_1]\!]_h^{I_u},\ldots,[\![t_1]\!]_h^{I_u}\big) \in P^{I_u} \\
\big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models \neg\phi &\iff \big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \not\models \phi \\
\big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models (\phi_1 \wedge \phi_2) &\iff \big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models \phi_1 \text{ and } \big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models \phi_2 \\
\big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models (\phi_1 \vee \phi_2) &\iff \big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models \phi_1 \text{ or } \big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models \phi_2 \\
\big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models \forall x\phi &\iff \text{for all } d \in \mathcal{D},\ \big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h[x \mapsto d], g, u \models \phi \\
\big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models \exists x\phi &\iff \text{there is } d \in \mathcal{D} \text{ s.t. } \big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h[x \mapsto d], g, u \models \phi \\
\big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models \bigcirc\phi &\iff \big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u+1 \models \phi \\
\big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models \bullet\phi &\iff u > 0 \text{ and } \big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u-1 \models \phi \\
\big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models \ominus\phi &\iff u = 0 \text{ or } \big[u > 0 \text{ and } \big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u-1 \models \phi\big] \\
\big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models \mu X\phi &\iff u \text{ belongs to the least fixpoint of the function} \\
&\qquad f_\phi : \wp(\mathbb{N}) \to \wp(\mathbb{N}),\ \text{defined as} \\
&\qquad f_\phi(S) = \big\{v \in \mathbb{N} \mid \big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g[X \mapsto S], v \models \phi\big\} \\
\big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g, u \models \nu X\phi &\iff u \text{ belongs to the greatest fixpoint of the function} \\
&\qquad f_\phi : \wp(\mathbb{N}) \to \wp(\mathbb{N}),\ \text{defined as} \\
&\qquad f_\phi(S) = \big\{v \in \mathbb{N} \mid \big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big), h, g[X \mapsto S], v \models \phi\big\}
\end{aligned}$$

We write $h[x \mapsto d]$ to denote the function that maps x to d and every $y \in \mathcal{V} \setminus \{x\}$ to $h(y)$. That is, $h[x \mapsto d]$ differs from h only at x. Similarly for $g[X \mapsto S]$.

The *truth set* $\|\phi\|_g^{\mathfrak{M},h}$ of a $\mathcal{FOTL}_{\mu\nu}$-formula ϕ is the set of moments where ϕ is true under the first-order temporal structure $\mathfrak{M}$, the individual variable assignment h, and the fixpoint variable assignment g. That is,

$$\|\phi\|_g^{\mathfrak{M},h} = \left\{ v \in \mathbb{N} \mid \mathfrak{M}, h, g, v \models \phi \right\} .$$

It should be noted that we are not done until we have shown that semantics is well-defined. We will continue to prove that the function f_ϕ for a fixpoint formula ${}^{\mu}_{\nu}X\phi$ has indeed least and greatest fixpoint. This is the case because of the requirement that the free occurrences of X in ϕ are positive. We can show the monotonicity of the function f_ϕ and argue by virtue of the Knaster-Tarski theorem [56] that f_ϕ has least and greatest fixpoint.

Proposition 4 (semantics is well-defined)**.** Fix a first-order temporal structure $\mathfrak{M} = \big(\langle \mathbb{N}, < \rangle, \mathcal{D}, I\big)$ and let ϕ be a $\mathcal{FOTL}_{\mu\nu}$-formula. Then, for any individual variable assignment $h : \mathcal{V} \to \mathcal{D}$, any fixpoint variable assignment $g : \wp(\mathbb{N}) \to \wp(\mathbb{N})$, and any moment $u \in \mathbb{N}$ the following hold.

(1) Semantics is defined for ϕ.

(2) For any fixpoint variable $X \in \mathcal{X}$ and any $A, B \in \wp(\mathbb{N})$ with $A \subseteq B$, we have that

  (a) if all free occurrences of X in ϕ are positive, then

$$\mathfrak{M}, h, g[X \mapsto A], u \models \phi \implies \mathfrak{M}, h, g[X \mapsto B], u \models \phi$$

  and

  (b) if all free occurrences of X in ϕ are negative, then

$$\mathfrak{M}, h, g[X \mapsto B], u \models \phi \implies \mathfrak{M}, h, g[X \mapsto A], u \models \phi .$$

Proof. By induction on the structure of ϕ.

- $\phi = p$. Fix h, g, u. (1) Clearly, semantics is defined. (2) Obvious, since truth for p only depends on p^{I_u}.
- $\phi = P(t_1, \ldots, t_n)$. Fix h, g, u. (1) Clearly, semantics is defined. (2) Obvious, since truth for $P(t_1, \ldots, t_n)$ depends on P^{I_u}.
- $\phi = Y$. Fix h, g, u. (1) Clearly, semantics is defined. (2) Fix $X \in \mathcal{X}$ and $A \subseteq B \subseteq \mathbb{N}$. If $Y \neq X$, then we observe that truth for Y depends only on $g(Y)$. If $Y = X$, then all free occurrences of X in ϕ are positive and $\mathfrak{M}, h, g[X \mapsto A], u \models \phi = X$ implies that $u \in A \subseteq B$ and hence $\mathfrak{M}, h, g[X \mapsto B], u \models X = \phi$.
- $\phi = \neg\phi_1$. Fix h, g, u. (1) By the inductive hypothesis (1), semantics is defined for ϕ_1. It follows that semantics is defined for $\neg\phi_1 = \phi$. (2) Fix $X \in \mathcal{X}$ and $A \subseteq B \subseteq \mathbb{N}$. (a) Suppose that all free occurrences of X in $\phi = \neg\phi_1$ are positive. Then, all free occurrences of X in ϕ_1 are negative. Suppose now that $\mathfrak{M}, h, g[X \mapsto A], u \models \phi = \neg\phi_1$, which means that $\mathfrak{M}, h, g[X \mapsto A], u \not\models \phi_1$. By the inductive hypothesis (contrapositive of 2b), $\mathfrak{M}, h, g[X \mapsto B], u \not\models \phi_1$ and hence $\mathfrak{M}, h, g[X \mapsto B], u \models \neg\phi_1 = \phi$. (b) Similarly.
- The cases $\phi = (\phi_1 \wedge \phi_2), (\phi_1 \vee \phi_2), \circ\phi_1, \bullet\phi_1, \ominus\phi_1$ are all easy.
- $\phi = \forall x\phi_1$. Fix h, g, u. (1) Easy. (2) Fix $X \in \mathcal{X}$ and $A \subseteq B \subseteq \mathbb{N}$. (a) Suppose that all free occurrences of X in $\forall x\phi_1$, and hence in ϕ_1, are positive and that $\mathfrak{M}, h, g[X \mapsto A], u \models \forall x\phi_1$. Assume to the contrary that $\mathfrak{M}, h, g[X \mapsto B], u \not\models \forall x\phi_1$. There is $d \in \mathcal{D}$ such that $\mathfrak{M}, h[x \mapsto d], g[X \mapsto B], u \not\models \phi_1$. From the contrapositive of inductive hypothesis (2a), we get that $\mathfrak{M}, h[x \mapsto d], g[X \mapsto A], u \not\models \phi_1$. Contradiction. (b) Similarly.
- $\phi = \exists x\phi_1$. Fix h, g, u. (1) Easy. (2) Fix $X \in \mathcal{X}$ and $A \subseteq B \subseteq \mathbb{N}$. (a) Suppose that all free occurrences of X in $\exists x\phi_1$, and hence in ϕ_1, are positive and that $\mathfrak{M}, h, g[X \mapsto A], u \models \exists x\phi_1$. There is $d \in \mathcal{D}$ such that $\mathfrak{M}, h[x \mapsto d], g[X \mapsto A], u \models \phi_1$. From the inductive hypothesis (2a), we have that $\mathfrak{M}, h[x \mapsto d], g[X \mapsto B], u \models \phi_1$, which implies that $\mathfrak{M}, h, g[X \mapsto B], u \models \exists x\phi_1$. (b) Similarly.
- $\phi = \mu Y\phi_1$. Fix h, g, u. (1) Consider the function

$$f : \wp(\mathbb{N}) \to \wp(\mathbb{N}) \qquad f(S) = \left\{ v \in \mathbb{N} \mid \mathfrak{M}, h, g[Y \mapsto S], v \models \phi_1 \right\} = \|\phi_1\|_{g[Y \mapsto S]}^{\mathfrak{M},h} .$$

By the inductive hypothesis (1), semantics is defined for ϕ_1 under $\mathfrak{M}, h, g[X \mapsto S]$ for any $S \subseteq \mathbb{N}$ and any moment v. So, f is a well-defined function. It remains to show that f has least fixpoint. Since $\mu Y \phi_1$ is a $\mathcal{FOTL}_{\mu\nu}$-formula, all free occurrences of Y in ϕ_1 are positive. This implies that f is monotonic. Let $S_1 \subseteq S_2 \subseteq \mathbb{N}$. We show that $f(S_1) \subseteq f(S_2)$. Let $v \in f(S_1)$, which means that $\mathfrak{M}, h, g[Y \mapsto S_1], v \models \phi_1$. By the inductive hypothesis (2a), we get that $\mathfrak{M}, h, g[Y \mapsto S_2], v \models \phi_1$ and hence $v \in f(S_2)$. Since $\wp(\mathbb{N})$ is a complete lattice with respect to set inclusion, we have by the Knaster-Tarski theorem that f has least fixpoint. It follows that semantics is defined for ϕ. (2) Fix $X \in \mathcal{X}$ and $A \subseteq B \subseteq \mathbb{N}$. If $X = Y$, then all occurrences of X in ϕ are bound and hence truth for ϕ is independent of the assignment to X. The interesting case is when $X \neq Y$. (a) Suppose that all free occurrences of X in ϕ, and hence in ϕ_1, are positive and that $\mathfrak{M}, h, g[X \mapsto A], u \models \phi = \mu Y \phi_1$. Consider the functions

$$f_A : \wp(\mathbb{N}) \to \wp(\mathbb{N}) \quad f_A(S) = \big\{v \in \mathbb{N} \mid \mathfrak{M}, h, g[X \mapsto A, Y \mapsto S], v \models \phi_1\big\} = \|\phi_1\|^{\mathfrak{M},h}_{g[X \mapsto A, Y \mapsto S]}$$
$$f_B : \wp(\mathbb{N}) \to \wp(\mathbb{N}) \quad f_B(S) = \big\{v \in \mathbb{N} \mid \mathfrak{M}, h, g[X \mapsto B, Y \mapsto S], v \models \phi_1\big\} = \|\phi_1\|^{\mathfrak{M},h}_{g[X \mapsto B, Y \mapsto S]} .$$

Notice that all free occurrences of Y in ϕ_1 are positive. From the inductive hypothesis (2a), we deduce that both f_A and f_B are monotonic and that for any $S \in \wp(\mathbb{N})$, $f_A(S) \subseteq f_B(S)$. By the Knaster-Tarski theorem, f_A has least fixpoint $\mathit{lfp}(f_A)$ and f_B has least fixpoint $\mathit{lfp}(f_B)$, where

$$\mathit{lfp}(f_A) = \bigcap \underbrace{\big\{S \in \wp(\mathbb{N}) \mid f_A(S) \subseteq S\big\}}_{\mathit{PREF}(f_A)} \qquad \mathit{lfp}(f_B) = \bigcap \underbrace{\big\{S \in \wp(\mathbb{N}) \mid f_B(S) \subseteq S\big\}}_{\mathit{PREF}(f_B)}$$

Notice that $\mathit{PREF}(f_B) \subseteq \mathit{PREF}(f_A)$

$$S \in \mathit{PREF}(f_B) \implies f_B(S) \subseteq S \implies f_A(S) \subseteq S \implies S \in \mathit{PREF}(f_A),$$

which implies that $\mathit{lfp}(f_A) \subseteq \mathit{lfp}(f_B)$. Therefore, $u \in \mathit{lfp}(f_A)$, from which it follows that $u \in \mathit{lfp}(f_B)$ and $\mathfrak{M}, h, g[X \mapsto B], u \models \mu Y \phi_1 = \phi$. (b) The case in which all free occurrences of X in ϕ are negative is similar. As before, we have that f_A and f_B are monotonic, since all free occurrences of Y in ϕ_1 are positive (inductive hypothesis 2a). The inductive hypothesis (2b) gives us that for any $S \in \wp(\mathbb{N})$, $f_B(S) \subseteq f_A(S)$. Therefore, $\mathit{lfp}(f_B) \subseteq \mathit{lfp}(f_A)$ and we are done.

- $\phi = \nu Y \phi_1$. Fix h, g, u. (1) We prove this part with arguments similar to the ones for $\phi = \mu Y \phi_1$. By the Knaster-Tarski theorem, a monotonic function on the complete lattice $\wp(\mathbb{N})$ has greatest fixpoint. (2) As before, we define functions f_A, f_B, which are both monotonic. By the Knaster-Tarski theorem, f_A has greatest fixpoint $\mathit{gfp}(f_A)$ and f_B has greatest fixpoint $\mathit{gfp}(f_B)$, where

$$\mathit{gfp}(f_A) = \bigcup \underbrace{\big\{S \in \wp(\mathbb{N}) \mid S \subseteq f_A(S)\big\}}_{\mathit{POSTF}(f_A)} \qquad \mathit{gfp}(f_B) = \bigcup \underbrace{\big\{S \in \wp(\mathbb{N}) \mid S \subseteq f_B(S)\big\}}_{\mathit{POSTF}(f_B)}$$

It is then easy to see, for example, that $\forall S \subseteq \mathbb{N}\big[f_A(S) \subseteq f_B(S)\big]$ implies that $\mathit{POSTF}(f_A) \subseteq \mathit{POSTF}(f_B)$ and hence $\mathit{gfp}(f_A) \subseteq \mathit{gfp}(f_B)$. ☐

Definition 5 (equivalence)**.** Let ϕ, ψ be $\mathcal{FOTL}_{\mu\nu}$-formulas. We say that ϕ and ψ are *equivalent*, and write $\phi \equiv \psi$, if for any first-order temporal structure $\mathfrak{M}$, any individual variable assignment h, any fixpoint variable assignment g, and any moment $u \in \mathbb{N}$,

$$\mathfrak{M}, h, g, u \models \phi \iff \mathfrak{M}, h, g, u \models \psi .$$

Equivalently, for any $\mathfrak{M}, h, g$, $\|\phi\|^{\mathfrak{M},h}_g = \|\psi\|^{\mathfrak{M},h}_g$.

Definition 6 (satisfiability, equisatisfiability)**.** We say that a $\mathcal{FOTL}_{\mu\nu}$-formula ϕ is *satisfiable* if there is a first-order temporal structure $\mathfrak{M}$, an individual variable assignment h, a fixpoint variable assignment g, and a moment $u \in \mathbb{N}$ such that $\mathfrak{M}, h, g, u \models \phi$. Two $\mathcal{FOTL}_{\mu\nu}$-formulas ϕ, ψ are called *equisatisfiable* if ϕ is satisfiable if and only if ψ is satisfiable.

Note that a fixpoint variable is given semantics similarly to a propositional variable. So, if we are given a $\mathcal{FOTL}_{\mu\nu}$-formula ϕ with free fixpoint variables $X_1, \ldots, X_k$, we can construct an equisatisfiable $\mathcal{FOTL}_{\mu\nu}$-fp-sentence ψ, which is the result of replacing each X_i by a propositional variable p_{X_i} that does not occur in ϕ. For example,

$$\phi = \bigcirc(\exists x Px \wedge X) \vee \mu Y(\forall x Rxx \vee Z \vee \bullet^5 Y) \quad \psi = \bigcirc(\exists x Px \wedge p_X) \vee \mu Y(\forall x Rxx \vee p_Z \vee \bullet^5 Y)$$

ϕ and ψ are equisatisfiable. Thus, from a $\mathcal{FOTL}_{\mu\nu}$-formula we can easily construct an equisatisfiable fp-sentence.

If a $\mathcal{FOTL}_{\mu\nu}$-formula ϕ has free individual variables $\mathsf{fvars}[\phi] = \{x_1, \ldots, x_k\}$, then it is satisfiable iff the dom-sentence $\exists x_1 \cdots \exists x_k \phi$ is satisfiable.

Definition 7 (substitution)**.** Let ϕ, ψ be $\mathcal{FOTL}_{\mu\nu}$-formulas and X be a fixpoint variable. We denote the formula that results from substituting ψ for X in ϕ by $[\phi]\{\psi/X\}$. More formally,

$$\begin{aligned}
[p]\{\psi/X\} &= p \\
[Y]\{\psi/X\} &= \begin{cases} \psi, & \text{if } Y = X \\ Y, & \text{if } Y \neq X \end{cases} \\
[P(t_1, \ldots, t_n)]\{\psi/X\} &= P(t_1, \ldots, t_n) \\
[\circ\phi]\{\psi/X\} &= \circ[\phi]\{\psi/X\} \text{ , where } \circ \in \{\neg, \bigcirc, \bullet, \ominus\} \\
[\phi_1 \otimes \phi_2]\{\psi/X\} &= [\phi_1]\{\psi/X\} \otimes [\phi_2]\{\psi/X\} \text{ , where } \otimes \in \{\wedge, \vee\} \\
[Qx\phi]\{\psi/X\} &= Qx[\phi]\{\psi/X\} \text{ , where } Q \in \{\forall, \exists\} \\
[fY\phi]\{\psi/X\} &= \begin{cases} fY\phi, & \text{if } Y = X \\ fY[\phi]\{\psi/X\}, & \text{if } Y \neq X \end{cases} \text{ , where } f \in \{\mu, \nu\}
\end{aligned}$$

For example, $[\exists x \mu Y(Rxy \vee (X \wedge \bigcirc Y))]\{\psi/X\} = \exists x \mu Y(Rxy \vee (\psi \wedge \bigcirc Y))$. We should note that, when substituting, it is possible to get a formula that is not well-formed, as the following example illustrates.

$$\phi = \mu Z(Py \vee Y \vee \bigcirc Z) \quad \psi = \exists y Rxy \wedge \neg Z \quad [\phi]\{\psi/Y\} = \mu Z[Py \vee (\exists y Rxy \wedge \neg Z) \vee \bigcirc Z]$$

Clearly, $[\phi]\{\psi/Y\}$ is not a $\mathcal{FOTL}_{\mu\nu}$-formula, because the first occurrence of Z is negative.

Lemma 8 (substitution-assignment lemma)**.** Let ϕ be a $\mathcal{FOTL}_{\mu\nu}$-formula. For any $\psi \in \mathcal{FOTL}_{\mu\nu}$, any first-order temporal structure $\mathfrak{M}$, any individual variable assignment h, any fixpoint variable assignment g, any fixpoint variable X, and any moment $u \in \mathbb{N}$, if no free variable (individual or fixpoint) in ψ gets bound in $[\phi]\{\psi/X\}$, then

$$\mathfrak{M}, h, g, u \models [\phi]\{\psi/X\} \iff \mathfrak{M}, h, g[X \mapsto \|\psi\|_g^{\mathfrak{M},h}], u \models \phi \ .$$

Proof. By induction on ϕ.

- The cases $\phi = p, P(t_1, \ldots, t_n), Y$ are trivial.
- The cases $\phi = \neg\phi_1, (\phi_1 \wedge \phi_2), (\phi_1 \vee \phi_2), \bigcirc\phi_1, \bullet\phi_1, \ominus\phi_1$ are easy.
- $\phi = \forall x \phi_1$. Fix ψ,$\mathfrak{M}$ with domain $\mathcal{D}$, h, g, X, u. Suppose that no free variable in ψ gets bound in $[\phi]\{\psi/X\}$ and that $\mathfrak{M}, h, g, u \models [\phi]\{\psi/X\} = [\forall x \phi_1]\{\psi/X\} = \forall x[\phi_1]\{\psi/X\}$. For contradiction, assume that
$$\mathfrak{M}, h, g[X \mapsto \|\psi\|_g^{\mathfrak{M},h}], u \not\models \phi = \forall x \phi_1 \ ,$$
which implies that there is $d \in \mathcal{D}$ such that
$$\mathfrak{M}, h[x \mapsto d], g[X \mapsto \|\psi\|_g^{\mathfrak{M},h}], u \not\models \phi_1 \ .$$
Observe that x does not appear free in ψ (if it did, then it would get bound in $[\phi]\{\psi/X\}$) and therefore the truth set of ψ is independent of the assignment to x. So, we have that
$$\mathfrak{M}, h[x \mapsto d], g[X \mapsto \|\psi\|_g^{\mathfrak{M},h[x \mapsto d]}], u \not\models \phi_1 \ .$$
Clearly, no free variable in ψ gets bound in $[\phi_1]\{\psi/X\}$. By the inductive hypothesis,
$$\mathfrak{M}, h[x \mapsto d], g, u \not\models [\phi_1]\{\psi/X\} \ .$$

Contradiction. The converse is shown using similar arguments.

- $\phi = \exists x \phi_1$. Fix ψ,$\mathfrak{M}$ with domain $\mathcal{D}$, h, g, X, u. Suppose that no free variable in ψ gets bound in $[\phi]\{\psi/X\}$ and that
$$\mathfrak{M}, h, g, u \models [\phi]\{\psi/X\} = [\exists x \phi_1]\{\psi/X\} = \exists x [\phi_1]\{\psi/X\} \ ,$$
which means that there is $d \in \mathcal{D}$ such that $\mathfrak{M}, h[x \mapsto d], g, u \models [\phi_1]\{\psi/X\}$. Notice that x does not appear free in ψ and that no free variable in ψ gets bound in $[\phi_1]\{\psi/X\}$. From the inductive hypothesis, we get that
$$\mathfrak{M}, h[x \mapsto d], g[X \mapsto \|\psi\|_g^{\mathfrak{M}, h[x \mapsto d]}], u \models \phi_1$$
and, since the truth set of ψ is independent of the assignment to x,
$$\mathfrak{M}, h[x \mapsto d], g[X \mapsto \|\psi\|_g^{\mathfrak{M},h}], u \models \phi_1 \ .$$
It follows that
$$\mathfrak{M}, h, g[X \mapsto \|\psi\|_g^{\mathfrak{M},h}], u \models \exists x \phi_1 = \phi \ .$$
We show the converse similarly.
- $\phi = \mu Y \phi_1$. Fix $\psi, \mathfrak{M}, h, g, X, u$. Suppose that no free variable in ψ gets bound in $[\phi]\{\psi/X\}$ and hence in $[\phi_1]\{\psi/X\}$. If $Y = X$, then $[\phi]\{\psi/X\} = \phi$ and the property follows easily. If $Y \neq X$, then $[\phi]\{\psi/X\} = [\mu Y \phi_1]\{\psi/X\} = \mu Y [\phi_1]\{\psi/X\}$. We define the functions
$$f_1(S) = \big\{ v \in \mathbb{N} \mid \mathfrak{M}, h, g[Y \mapsto S], v \models [\phi_1]\{\psi/X\} \big\}$$
$$f_2(S) = \big\{ v \in \mathbb{N} \mid \mathfrak{M}, h, g[X \mapsto \|\psi\|_g^{\mathfrak{M},h}, Y \mapsto S], v \models \phi_1 \big\}$$
Observe that Y does not occur free in ψ and therefore
$$g[X \mapsto \|\psi\|_g^{\mathfrak{M},h}, Y \mapsto S] = g[Y \mapsto S][X \mapsto \|\psi\|_{g[Y \mapsto S]}^{\mathfrak{M},h}] \ .$$
From the inductive hypothesis, we easily deduce that for any $S \subseteq \mathbb{N}$, $f_1(S) = f_2(S)$. It follows that $\mathit{lfp}(f_1) = \mathit{lfp}(f_2)$, from which we get that
$$\begin{aligned} \mathfrak{M}, h, g, u \models [\phi]\{\psi/X\} = \mu Y [\phi_1]\{\psi/X\} \iff u \in \mathit{lfp}(f_1) \\ \iff u \in \mathit{lfp}(f_2) \iff \mathfrak{M}, h, g[X \mapsto \|\psi\|_g^{\mathfrak{M},h}], u \models \mu Y \phi_1 = \phi \ . \end{aligned}$$
- The case $\phi = \nu Y \phi_1$ is similar to the previous one. □

Lemma 9 (fixpoint unfolding)**.**

(1) Let $\mu X \phi$ be a $\mathcal{FOTL}_{\mu\nu}$-formula. If no free variable in $\mu X \phi$ gets bound in $[\phi]\{\mu X \phi/X\}$, then $\mu X \phi \equiv [\phi]\{\mu X \phi/X\}$.

(2) Let $\nu X \phi$ be a $\mathcal{FOTL}_{\mu\nu}$-formula. If no free variable in $\nu X \phi$ gets bound in $[\phi]\{\nu X \phi/X\}$, then $\nu X \phi \equiv [\phi]\{\nu X \phi/X\}$.

Proof. (1) Fix $\mathfrak{M}, h, g, u$. We define $f(S) = \|\phi\|_{g[X \mapsto S]}^{\mathfrak{M},h}$ and observe that $\mathit{lfp}(f) = \|\mu X \phi\|_g^{\mathfrak{M},h}$. We have that
$$\begin{aligned} \mathfrak{M}, h, g, u \models \mu X \phi \iff u \in \mathit{lfp}(f) = f(\mathit{lfp}(f)) \iff u \in \|\phi\|_{g[X \mapsto \mathit{lfp}(f)]}^{\mathfrak{M},h} \iff \\ \mathfrak{M}, h, g[X \mapsto \mathit{lfp}(f)], u \models \phi \iff \mathfrak{M}, h, g[X \mapsto \|\mu X \phi\|_g^{\mathfrak{M},h}], u \models \phi \overset{\text{Lemma 8}}{\iff} \\ \mathfrak{M}, h, g, u \models [\phi]\{\mu X \phi/X\} \ . \end{aligned}$$
(2) Similarly to (1). □

4. Signatures

Given a monotonic function f over subsets of a set, the Knaster-Tarski theorem [56] gives the least fixpoint of f, which we denote by $\mathit{lfp}(f)$, in terms of a transfinite sequence

$$\mu_0(f), \mu_1(f), \mu_2(f), \ldots, \mu_\omega(f), \mu_{\omega+1}(f), \mu_{\omega+2}(f), \ldots$$

that ranges over the class of ordinals. This sequence is increasing and we can say intuitively that at every step it gives us a better approximation of the least fixpoint. Similarly, the greatest fixpoint of f, denoted by $\mathit{gfp}(f)$, can be given in terms of a transfinite sequence

$$\nu_0(f), \nu_1(f), \nu_2(f), \ldots, \nu_\omega(f), \nu_{\omega+1}(f), \nu_{\omega+2}(f), \ldots$$

which is decreasing. Each step in the above sequences corresponds to an application of the function f.

The notion of signature, introduced later in this section, captures the intuition that in order to verify that a formula is satisfied at a specific moment, we only need to perform a certain "amount" of "iterations" (these iterations correspond to least fixpoint unfoldings) for the least fixpoints involved in the formula. Consider, for example, the least fixpoint formula $\phi = \mu X(\ominus\bot \vee \bullet^3 X)$, which is true at the moments $0, 3, 6, 9, 12, \ldots$

$$\begin{array}{ccccccccccccc}
\phi & \neg\phi & \neg\phi & \phi & \neg\phi & \neg\phi & \phi & \neg\phi & \neg\phi & \phi & \neg\phi & \neg\phi & \\
\bullet & \circ & \circ & \bullet & \circ & \circ & \bullet & \circ & \circ & \bullet & \circ & \circ & \longrightarrow \mathbb{N} \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 &
\end{array}$$

The truth set of ϕ is the least fixpoint of the function $f(S) = \{0\} \cup \{u + 3 \mid u \in S\}$. The sequence of approximations of $\mathit{lfp}(f)$ is

$$\emptyset, \{0\}, \{0,3\}, \{0,3,6\}, \{0,3,6,9\}, \{0,3,6,9,12\}, \{0,3,6,9,15\}, \ldots$$

which means that at every moment $3u$ that is in the truth set of ϕ, we need $(u+1)$ iterations to verify that $3u$ is indeed in the least fixpoint.

First, we will discuss how we can obtain positive form by "pushing down" the negations. We continue with a formal definition of fixpoint approximants. Then, we proceed to show some lemmas that give us confidence that our definitions have sufficiently captured our intuition about approximants. For example, we will show that if we "increase the amount of least fixpoint iterations", we get a wider truth set. Subsequently, the concept of signature is defined. We speak of μ-signatures and their dual ν-signatures. We conclude with an important proposition, the main point of which is that μ-signature decreases when a least fixpoint is unfolded and does not increase in the rest of the cases.

4.1. Positive Form.

The operators of the language are sufficient to transform any $\mathcal{FOTL}_{\mu\nu}$-formula to an equivalent one in which all negations occur only in front of propositional variables of fixpoint variables. We then say that the formula is in *positive form*. Apart from the well-known equivalences

$$\neg\neg\phi \equiv \phi \qquad \neg(\phi_1 \wedge \phi_2) \equiv \neg\phi_1 \vee \neg\phi_2 \qquad \neg\forall x\phi \equiv \exists x\neg\phi$$
$$\neg(\phi_1 \vee \phi_2) \equiv \neg\phi_1 \wedge \neg\phi_2 \qquad \neg\exists x\phi \equiv \forall x\neg\phi$$

we can easily prove the following.

$$\neg\bigcirc\phi \equiv \bigcirc\neg\phi \qquad \neg\bullet\phi \equiv \ominus\neg\phi \qquad \neg\ominus\phi \equiv \bullet\neg\phi$$

It remains to show that we can push negations in when they occur in front of fixpoint formulas.

Lemma 10 (pushing negation in through fixpoints)**.** For any $\mathcal{FOTL}_{\mu\nu}$-formula ϕ and any fixpoint variable X with all free occurrences of X in ϕ being positive,

$$\neg\mu X\phi \equiv \nu X\neg[\phi]\{\neg X/X\} \qquad \neg\nu X\phi \equiv \mu X\neg[\phi]\{\neg X/X\}\ .$$

Proof. Fix a first-order temporal structure $\mathfrak{M}$, an individual variable assignment h, a fixpoint variable assignment g, and a moment $u \in \mathbb{N}$. We define the functions

$$f_1 : \wp(\mathbb{N}) \to \wp(\mathbb{N}) \qquad f_1(S) = \{v \in \mathbb{N} \mid \mathfrak{M}, h, g[X \mapsto S], v \models \phi\}$$
$$f_2 : \wp(\mathbb{N}) \to \wp(\mathbb{N}) \qquad f_2(S) = \{v \in \mathbb{N} \mid \mathfrak{M}, h, g[X \mapsto S], v \models \neg[\phi]\{\neg X/X\}\} \ .$$

We observe that

$$\|\neg\mu X\phi\|_g^{\mathfrak{M},h} = \mathbb{N} \setminus \|\mu X\phi\|_g^{\mathfrak{M},h} = \mathbb{N} \setminus \mathit{lfp}(f_1) \qquad \|\nu X\neg[\phi]\{\neg X/X\}\|_g^{\mathfrak{M},h} = \mathit{gfp}(f_2) \ ,$$

and that for any $S \in \wp(\mathbb{N})$,

$$\begin{aligned} f_2(S) &= \mathbb{N} \setminus \{v \in \mathbb{N} \mid (\mathbb{N},<), h, g[X \mapsto S], v \models [\phi]\{\neg X/X\}\} \\ &= \mathbb{N} \setminus \{v \in \mathbb{N} \mid (\mathbb{N},<), h, g[X \mapsto \|\neg X\|^h_{g[X\mapsto S]}], v \models \phi\} && \text{(Lemma 8)} \\ &= \mathbb{N} \setminus \{v \in \mathbb{N} \mid (\mathbb{N},<), h, g[X \mapsto \mathbb{N} \setminus S], v \models \phi\} \\ &= \mathbb{N} \setminus f_1(\mathbb{N} \setminus S) \ . \end{aligned}$$

It also holds that for any $S \in \wp(\mathbb{N})$, $f_1(S) = \mathbb{N} \setminus f_2(\mathbb{N} \setminus S)$. Let $S \subseteq \mathbb{N}$ and $S^c = \mathbb{N} \setminus S$. It follows that $f_2(S^c) = \mathbb{N} \setminus f_1(\mathbb{N} \setminus S^c) = \mathbb{N} \setminus f_1(S)$, which implies that $f_1(S) = \mathbb{N} \setminus f_2(S^c) = \mathbb{N} \setminus f_2(\mathbb{N} \setminus S)$.

We have that $\mathit{gfp}(f_2) = f_2(\mathit{gfp}(f_2)) = \mathbb{N} \setminus f_1(\mathbb{N} \setminus \mathit{gfp}(f_2))$, which implies that $\mathbb{N} \setminus \mathit{gfp}(f_2) = f_1(\mathbb{N} \setminus \mathit{gfp}(f_2))$. It follows that $\mathbb{N} \setminus \mathit{gfp}(f_2)$ is a fixpoint of f_1 and hence $\mathit{lfp}(f_1) \subseteq \mathbb{N} \setminus \mathit{gfp}(f_2)$. Similarly, $\mathit{lfp}(f_1) = f_1(\mathit{lfp}(f_1)) = \mathbb{N} \setminus f_2(\mathbb{N} \setminus \mathit{lfp}(f_1))$, which means that $\mathbb{N} \setminus \mathit{lfp}(f_1) = f_2(\mathbb{N} \setminus \mathit{lfp}(f_1))$. Therefore, $\mathbb{N} \setminus \mathit{lfp}(f_1)$ is a fixpoint of f_2 and hence $\mathbb{N} \setminus \mathit{lfp}(f_1) \subseteq \mathit{gfp}(f_2)$. So, $\mathbb{N} \setminus \mathit{gfp}(f_2) \subseteq \mathit{lfp}(f_1)$. We have that $\mathit{lfp}(f_1) = \mathbb{N} \setminus \mathit{gfp}(f_2)$ and we are done. We prove the equivalence $\neg\nu X\phi \equiv \mu X\neg[\phi]\{\neg X/X\}$ with similar arguments. ☐

Definition 11 ($\mathcal{FOTL}^{\mathsf{pos}}_{\mu\nu}$)**.** We define the set of $\mathcal{FOTL}^{\mathsf{pos}}_{\mu\nu}$-formulas to be the smallest subset of $\Sigma^\star$ that satisfies the following conditions.

- $\mathcal{FOTL}_{\mu\nu}$ includes $\mathcal{P}_0$ and $\{\neg p \mid p \in \mathcal{P}_0\}$.
- $\mathcal{FOTL}_{\mu\nu}$ includes $\mathcal{X}$ and $\{\neg X \mid X \in \mathcal{X}\}$.
- For any predicate symbol P, if $t_1, \ldots, t_n \in \mathcal{T}$ ($n = \mathsf{ar}[P]$), then $P(t_1, \ldots, t_n), \neg P(t_1, \ldots, t_n) \in \mathcal{FOTL}^{\mathsf{pos}}_{\mu\nu}$.
- For any unary operator $\circ \in \{\bigcirc, \bullet, \ominus\}$, if $\phi \in \mathcal{FOTL}^{\mathsf{pos}}_{\mu\nu}$, then $\circ\phi \in \mathcal{FOTL}^{\mathsf{pos}}_{\mu\nu}$.
- For any binary operator $\otimes \in \{\wedge, \vee\}$, if $\phi_1, \phi_2 \in \mathcal{FOTL}^{\mathsf{pos}}_{\mu\nu}$, then $(\phi_1 \otimes \phi_2) \in \mathcal{FOTL}^{\mathsf{pos}}_{\mu\nu}$.
- For any quantifier $Q \in \{\forall, \exists\}$, if $x \in \mathcal{V}$ and $\phi \in \mathcal{FOTL}^{\mathsf{pos}}_{\mu\nu}$, then $Qx\phi \in \mathcal{FOTL}^{\mathsf{pos}}_{\mu\nu}$.
- For any fixpoint operator $f \in \{\mu, \nu\}$, if X is a fixpoint variable, $\phi \in \mathcal{FOTL}^{\mathsf{pos}}_{\mu\nu}$, and all free occurrences of X in ϕ are positive, then $fX\phi \in \mathcal{FOTL}^{\mathsf{pos}}_{\mu\nu}$.

Definition 12 ($\mathsf{pos}[\cdot], \mathsf{neg}[\cdot]$)**.** We define the function $\mathsf{pos}[\cdot] : \mathcal{FOTL}_{\mu\nu} \to \mathcal{FOTL}^{\mathsf{pos}}_{\mu\nu}$, which converts a $\mathcal{FOTL}_{\mu\nu}$-formula to an equivalent one in positive form.

$$\begin{aligned}
\mathsf{pos}[\neg p] &= \neg p & \mathsf{pos}[p] &= p \\
\mathsf{pos}[\neg X] &= \neg X & \mathsf{pos}[X] &= X \\
\mathsf{pos}[\neg P(t_1, \ldots, t_n)] &= \neg P(t_1, \ldots, t_n) & \mathsf{pos}[P(t_1, \ldots, t_n)] &= P(t_1, \ldots, t_n) \\
\mathsf{pos}[\neg\neg\phi] &= \mathsf{pos}[\phi] \\
\mathsf{pos}[\neg(\phi_1 \wedge \phi_2)] &= \mathsf{pos}[\neg\phi_1] \vee \mathsf{pos}[\neg\phi_2] & \mathsf{pos}[\phi_1 \wedge \phi_2] &= \mathsf{pos}[\phi_1] \wedge \mathsf{pos}[\phi_2] \\
\mathsf{pos}[\neg(\phi_1 \vee \phi_2)] &= \mathsf{pos}[\neg\phi_1] \wedge \mathsf{pos}[\neg\phi_2] & \mathsf{pos}[\phi_1 \vee \phi_2] &= \mathsf{pos}[\phi_1] \vee \mathsf{pos}[\phi_2] \\
\mathsf{pos}[\neg\forall x\phi] &= \exists x\mathsf{pos}[\neg\phi] & \mathsf{pos}[\forall x\phi] &= \forall x\mathsf{pos}[\phi] \\
\mathsf{pos}[\neg\exists x\phi] &= \forall x\mathsf{pos}[\neg\phi] & \mathsf{pos}[\exists x\phi] &= \exists x\mathsf{pos}[\phi] \\
\mathsf{pos}[\neg\bigcirc\phi] &= \bigcirc\mathsf{pos}[\neg\phi] & \mathsf{pos}[\bigcirc\phi] &= \bigcirc\mathsf{pos}[\phi] \\
\mathsf{pos}[\neg\bullet\phi] &= \ominus\mathsf{pos}[\neg\phi] & \mathsf{pos}[\bullet\phi] &= \bullet\mathsf{pos}[\phi] \\
\mathsf{pos}[\neg\ominus\phi] &= \bullet\mathsf{pos}[\neg\phi] & \mathsf{pos}[\ominus\phi] &= \ominus\mathsf{pos}[\phi] \\
\mathsf{pos}[\neg\mu X\phi] &= \nu X\mathsf{pos}[\neg[\phi]\{\neg X/X\}] & \mathsf{pos}[\mu X\phi] &= \mu X\mathsf{pos}[\phi] \\
\mathsf{pos}[\neg\nu X\phi] &= \mu X\mathsf{pos}[\neg[\phi]\{\neg X/X\}] & \mathsf{pos}[\nu X\phi] &= \nu X\mathsf{pos}[\phi]
\end{aligned}$$

We also define $\mathsf{neg}[\phi] = \mathsf{pos}[\neg\phi]$. Consider a formula ϕ in positive form. The positive form of its negation $\mathsf{neg}[\phi]$ results from ϕ by replacing $\wedge$ by $\vee$, $\vee$ by $\wedge$, $\forall$ by $\exists$, $\exists$ by $\forall$, $\bullet$ by $\ominus$, $\ominus$ by $\bullet$, μ by ν, ν by μ, p by $\neg p$, $\neg p$ by p, $P(t_1, \ldots, t_n)$ by $\neg P(t_1, \ldots, t_n)$, and $\neg P(t_1, \ldots, t_n)$ by $P(t_1, \ldots, t_n)$. For example,

$$\phi = \mu X\Big\{\bullet\exists x Px \vee \big[\forall y Ryy \wedge \bigcirc\mu Y(X \vee (Qz \wedge \bigcirc Y))\big]\Big\} \wedge \nu W\big[\nu Z(\neg p \wedge \bigcirc Z) \wedge \ominus^3 W\big]$$

$$\mathsf{neg}[\phi] = \nu X\Big\{\ominus\forall\neg Px \wedge \big[\exists\neg Ryy \vee \bigcirc\nu Y(X \wedge (\neg Qz \vee \bigcirc Y))\big]\Big\} \vee \mu W\big[\mu Z(p \vee \bigcirc Z) \vee \bullet^3 W\big] \ .$$

It is easy to see that $\mathsf{neg}[\mathsf{neg}[\phi]] = \phi$, which can be expressed equivalently as $\psi = \mathsf{neg}[\phi] \iff \phi = \mathsf{neg}[\psi]$. In order to cut down on the clutter of the brackets, we will sometimes write $+\chi$ to mean $\mathsf{pos}[\chi]$ and $\sim\chi$ to mean $\mathsf{neg}[\chi]$.

Lemma 13. Let ϕ be a $\mathcal{FOTL}_{\mu\nu}$-formula. Then, for any $\psi \in \mathcal{FOTL}_{\mu\nu}$ and any $X \in \mathcal{X}$,

(1) if all free occurrences of X in ϕ are positive, then

$$+[\phi]\{\psi/X\} = [+\phi]\{+\psi/X\} \qquad\qquad \sim[\phi]\{\psi/X\} = [\sim[\phi]\{\neg X/X\}]\{\sim\psi/X\}$$

(2) if all free occurrences of X in ϕ are negative, then

$$+[\phi]\{\psi/X\} = [+[\phi]\{\neg X/X\}]\{\sim\psi/X\} \qquad\qquad \sim[\phi]\{\psi/X\} = [\sim\phi]\{+\psi/X\} \ .$$

Proof. By induction on ϕ. □

4.2. **Fixpoint Approximants.** Let $\mu X\phi$ be a $\mathcal{FOTL}_{\mu\nu}$-formula that is true at u under $\mathfrak{M}, h, g$. That is, $\mathfrak{M}, h, g, u \models \mu X\phi$ or equivalently, u belongs to the least fixpoint of the monotonic function

$$f(S) = \big\{v \in \mathbb{N} \mid \mathfrak{M}, h, g[X \mapsto S], v \models \phi\big\} = \|\phi\|^{\mathfrak{M},h}_{g[X\mapsto S]} \ .$$

By the Knaster-Tarski theorem, the least fixpoint of f is equal to

$$\mathit{lfp}(f) = \sup_{\alpha\in\mathbf{Ord}} \mu_\alpha(f) = \bigcup_{\alpha\in\mathbf{Ord}} \mu_\alpha(f) \ ,$$

where **Ord** is the proper class of ordinals and $\mu_\alpha(f)$ is defined as

$$\mu_0(f) = \emptyset \qquad \mu_{\alpha+1}(f) = f(\mu_\alpha(f)) \qquad \mu_\lambda(f) = \sup_{\alpha<\lambda} \mu_\alpha(f) = \bigcup_{\alpha<\lambda} \mu_\alpha(f), \text{ for limit ordinal } \lambda$$

It follows that $u \in \mathit{lfp}(f)$ and hence there is some ordinal α such that $u \in \mu_\alpha(f)$. It is easy to show that if $\alpha < \beta$, then $\mu_\alpha(f) \subseteq \mu_\beta(f)$.

Consider now a $\mathcal{FOTL}_{\mu\nu}$-formula $\nu X\phi$ that is false at u under $\mathfrak{M}, h, g$. That is, u does not belong to the greatest fixpoint of f, which is equal to

$$\mathit{gfp}(f) = \inf_{\alpha\in\mathbf{Ord}} \nu_\alpha(f) = \bigcap_{\alpha\in\mathbf{Ord}} \nu_\alpha(f) \ .$$

We deduce that there is an ordinal α such that $u \notin \nu_\alpha(f)$. $\nu_\alpha(f)$ is defined as follows.

$$\nu_0(f) = \mathbb{N} \qquad \nu_{\alpha+1}(f) = f(\nu_\alpha(f)) \qquad \nu_\lambda(f) = \inf_{\alpha<\lambda} \nu_\alpha(f) = \bigcap_{\alpha<\lambda} \nu_\alpha(f), \text{ for limit ordinal } \lambda$$

If $\alpha < \beta$, then $\nu_\beta(f) \subseteq \nu_\alpha(f)$.

Definition 14 (annotated fixpoint operators)**.** We extend the syntax so that we also have annotated fixpoint operators μ_α and ν_α for any ordinal α. We then get the set of formulas $\mathcal{FOTL}_{\mu\nu\alpha}$ and the corresponding set $\mathcal{FOTL}^{\mathsf{pos}}_{\mu\nu\alpha}$ of $\mathcal{FOTL}_{\mu\nu\alpha}$-formulas in positive form. Semantics is defined as one would expect.

$$\begin{aligned}
\mathfrak{M}, h, g, u \models \mu_\alpha X\phi \iff{}& u \in \mu_\alpha(f_\phi), \text{ where } f_\phi : \wp(\mathbb{N}) \to \wp(\mathbb{N}) \text{ is defined as} \\
& f_\phi(S) = \big\{v \in \mathbb{N} \mid \mathfrak{M}, h, g[X \mapsto S], v \models \phi\big\} = \|\phi\|^{\mathfrak{M},h}_{g[X\mapsto S]} \\
\mathfrak{M}, h, g, u \models \nu_\alpha X\phi \iff{}& u \in \nu_\alpha(f_\phi), \text{ where } f_\phi : \wp(\mathbb{N}) \to \wp(\mathbb{N}) \text{ is defined as} \\
& f_\phi(S) = \big\{v \in \mathbb{N} \mid \mathfrak{M}, h, g[X \mapsto S], v \models \phi\big\} = \|\phi\|^{\mathfrak{M},h}_{g[X\mapsto S]}
\end{aligned}$$

Definition 15 (positive normal form)**.** We say that a $\mathcal{FOTL}_{\mu\nu}$-formula ϕ is in *positive normal form* if it is positive and all fixpoint formulas in $\mathsf{sbf}[\phi]$ bind different fixpoint variables. For example,

$$\phi = \underbrace{\mu X(\exists x Px \vee \bigcirc X)}_{\phi_1} \wedge \bullet^2 \underbrace{\nu Y(p \wedge \ominus^2 Y)}_{\phi_2} \wedge \overbrace{\mu Z\big[\underbrace{\mu X(\exists x Px \vee \bigcirc X)}_{\phi_1} \vee \bullet^5 Z\big]}^{\phi_3}$$

all the fixpoint subformulas of ϕ (i.e. ϕ_1, ϕ_2, and ϕ_3) bind the distinct fixpoint variables X, Y, Z respectively and hence ϕ is in positive normal form. The formula

$$\psi = \underbrace{\mu Y(\forall x \neg Qx \vee \bigcirc^3 Y)}_{\psi_1} \wedge \ominus \bigcirc \bullet \underbrace{\nu Y(\exists y \forall z Ryz \wedge \ominus^2 Y)}_{\psi_2} ,$$

however, is not in positive normal form, because its fixpoint subformulas ψ_1, ψ_2 bind the same fixpoint variable, namely Y.

It is easy to see that renaming a bound fixpoint variable does not change semantics. For example, the formulas $\phi = \exists x \mu X(Rxx \vee \bigcirc X)$ and $\phi = \exists x \mu Y(Rxx \vee \bigcirc Y)$ are equivalent. So, for any $\mathcal{FOTL}_{\mu\nu}$-formula ϕ there is an equivalent one in positive normal form, which can be easily constructed from ϕ by applying the $\mathsf{pos}[\cdot]$ function and renaming the bound fixpoint variables appropriately. Some important properties follow.

(1) Let ${}^{\mu}_{\nu}X\phi$ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-formula. Then, $[\phi]\{{}^{\mu}_{\nu}X\phi/X\}$ is *not* necessarily a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-formula. For example,

$$\begin{aligned} \nu X\phi &= \nu X \mu Y[(Px \vee \bullet Y) \wedge \bigcirc X] && \text{in p.n.f.} \\ [\phi]\{\nu X\phi/X\} &= \mu Y\big\{(Px \vee \bullet Y) \wedge \bigcirc \nu X \mu Y[(Px \vee \bullet Y) \wedge \bigcirc X]\big\} && \text{not in p.n.f.} \end{aligned}$$

However, we can rectify this by renaming appropriately some bound fixpoint variables, while preserving semantics. For the previous example, we would rename the bound variable Y in ϕ to Z and thus

$$[\dddot{\phi}]\{\nu X\phi/X\} = \mu Z\big\{(Px \vee \bullet Z) \wedge \bigcirc \nu X \mu Y[(Px \vee \bullet Y) \wedge \bigcirc X]\big\} ,$$

where $\dddot{\phi}$ is equal to ϕ modulo the name of one bound variable. It is clear that $[\dddot{\phi}]\{\nu X\phi/X\}$ is in positive normal form.

(2) Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-formula. Then, μ-$\mathsf{vars}[\phi]$ and ν-$\mathsf{vars}[\phi]$ are disjoint.

(3) Let $\mu X\phi$ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-formula. Then, $X \notin \mu$-$\mathsf{vars}[\phi]$.

(4) Let $\nu X\phi$ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-formula. Then, $X \notin \nu$-$\mathsf{vars}[\phi]$.

Definition 16 (annotation)**.** We say that a partial function from the set of fixpoint variables to the class of ordinals is an *annotation*. An annotation $a : \mathcal{X} \to \mathbf{Ord}$ induces a function from $\mathcal{FOTL}_{\mu\nu}$ to $\mathcal{FOTL}_{\mu\nu\alpha}$ as follows.

$$\begin{aligned} \mathsf{ann}[p, a] &= p \\ \mathsf{ann}[X, a] &= X \\ \mathsf{ann}[P(t_1, \dots, t_n), a] &= P(t_1, \dots, t_n) \\ \mathsf{ann}[\circ\phi, a] &= \circ\mathsf{ann}[\phi, a] \text{ , where } \circ \in \{\neg, \bigcirc, \bullet, \ominus\} \\ \mathsf{ann}[\phi_1 \otimes \phi_2, a] &= \mathsf{ann}[\phi_1, a] \otimes \mathsf{ann}[\phi_2, a] \text{ , where } \otimes \in \{\wedge, \vee\} \\ \mathsf{ann}[Qx\phi, a] &= Qx\mathsf{ann}[\phi, a] \text{ , where } Q \in \{\forall, \exists\} \\ \mathsf{ann}[fX\phi, a] &= \begin{cases} f_{a(X)}X\mathsf{ann}[\phi, a], & \text{if } a(X) \text{ is defined} \\ fX\mathsf{ann}[\phi, a], & \text{if } a(X) \text{ is undefined} \end{cases} \text{ , where } f \in \{\mu, \nu\} \end{aligned}$$

For example,

$$\begin{aligned}
\phi &= \mu X\big[\mu Y(\exists x Px \vee \bullet^{10} Y) \vee \bigcirc^3 X\big] \wedge \forall y \nu Z(\neg Ryy \wedge \ominus^4 Z) \wedge \mu Y(p \vee \bullet^2 Y) \\
a &= \{X \mapsto 5, Y \mapsto \omega\} \\
\mathsf{ann}[\phi, a] &= \mu_5 X\big[\mu_\omega Y(\exists x Px \vee \bullet^{10} Y) \vee \bigcirc^3 X\big] \wedge \forall y \nu Z(\neg Ryy \wedge \ominus^4 Z) \wedge \mu_\omega Y(p \vee \bullet^2 Y)
\end{aligned}$$

Note that $\mathsf{ann}[\phi, a]$ depends only on the values of a for $\mu\nu$-$\mathsf{vars}[\phi]$. That is, any two annotations a, b that have the same values for all the fixpoint variables in $\mu\nu$-$\mathsf{vars}[\phi]$ annotate the formula ϕ in exactly the same way.

Definition 17 (ordering annotations)**.** Let $a : \mathcal{X} \to \mathbf{Ord}$, $b : \mathcal{X} \to \mathbf{Ord}$ be annotations. We define the partial order $\leq$ and the strict partial order $<$ on annotations as follows.

$$\begin{aligned}
a \leq b &\overset{\text{def}}{\iff} \mathit{Domain}(a) \subseteq \mathit{Domain}(b) \;\& \\
&\qquad \text{for any } X \in \mathit{Domain}(a),\ a(X) \leq b(X) \\
a < b &\overset{\text{def}}{\iff} a \leq b \;\&\; \text{there is } Y \in \mathit{Domain}(a) \text{ s.t. } a(Y) < b(Y)\ .
\end{aligned}$$

Consider now the annotations that are defined on a specified finite set of fixpoint variables $V = \{X_0, X_1, \ldots, X_k\}$, which is linearly ordered as $X_0 < X_1 < \ldots < X_k$. They can be easily well-ordered lexicographically. That is, for any $a : V \to \mathbf{Ord}$, $b : V \to \mathbf{Ord}$,

$$a <_V b \overset{\text{def}}{\iff} a(X_0)a(X_1)\ldots a(X_k) <_{\text{lex}} b(X_0)b(X_1)\ldots b(X_k)\ .$$

It is easy to see that for any annotations a, b defined on V, $a < b \implies a <_V b$. We write $a \leq_V b$ to mean ($a <_V b$ or $a = b$).

Lemma 18 (annotation and substitution)**.** Let ϕ, ψ be $\mathcal{FOTL}_{\mu\nu}$-formulas, X be a fixpoint variable, and a an annotation. Then,

$$\mathsf{ann}[[\phi]\{\psi/X\}, a] = [\mathsf{ann}[\phi, a]]\{\mathsf{ann}[\psi, a]/X\}\ .$$

Proof. An easy induction on ϕ. □

Lemma 19. $\mathsf{pos}[\mathsf{ann}[\phi, a]] = \mathsf{ann}[\mathsf{pos}[\phi], a]$.

Proof. Easy. □

Definition 20 (μ-annotation, ν-annotation)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-formula and $a : \mathcal{X} \to \mathbf{Ord}$. We say that a is a *μ-annotation for ϕ* if it is defined at all fixpoint variables in μ-$\mathsf{vars}[\phi]$ and at no fixpoint variables in ν-$\mathsf{vars}[\phi]$. The idea is that a μ-annotation only annotates least fixpoint subformulas. Similarly, we say that a is a *ν-annotation for ϕ* if it is defined on ν-$\mathsf{vars}[\phi]$ and undefined on μ-$\mathsf{vars}[\phi]$. Thus, a ν-annotation only annotates greatest fixpoint subformulas. These definitions are meaningful, because (as noted in Definition 15) for a formula ϕ in positive normal form, μ-$\mathsf{vars}[\phi] \cap \nu$-$\mathsf{vars}[\phi] = \emptyset$.

Observe all ν (μ) operators in ϕ become μ (ν) operators in $\mathsf{neg}[\phi]$. So, a ν-annotation for ϕ is a μ-annotation for $\mathsf{neg}[\phi]$. Similarly, a μ-annotation for ϕ is a ν-annotation for $\mathsf{neg}[\phi]$.

Lemma 21 (increasing μ-annotation)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-formula. Then, for any μ-annotations a, b for ϕ with $a \leq b$, any first-order temporal structure $\mathfrak{M}$, any individual variable assignment h, any fixpoint variable assignment g, and any moment $u \in \mathbb{N}$,

$$\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi, a] \implies \mathfrak{M}, h, g, u \models \mathsf{ann}[\phi, b]\ .$$

Proof. By induction on ϕ.

- The cases $\phi = p, \neg p, X, \neg X, P(t_1, \ldots, t_n), \neg P(t_1, \ldots, t_n)$ are trivial.
- $\phi = (\phi_1 \wedge \phi_2)$. Fix a, b with $a \leq b$, and $\mathfrak{M}, h, g, u$. Suppose that $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi, a]$, which implies that $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi_1, a]$ and $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi_2, a]$. By the inductive hypothesis, $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi_1, b]$ and $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi_2, b]$. Therefore, $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi_1, b] \wedge \mathsf{ann}[\phi_2, b] = \mathsf{ann}[\phi_1 \wedge \phi_2, b]$.
- The case $\phi = (\phi_1 \vee \phi_2)$ is dealt with similar arguments.

- $\phi = \forall x\phi_1$. Fix a, b with $a \le b$, $\mathfrak{M}$ with domain $\mathcal{D}$, and h, g, u. Suppose that $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi, a] = \mathsf{ann}[\forall x\phi_1, a] = \forall x\mathsf{ann}[\phi_1, a]$ and assume for contradiction that $\mathfrak{M}, h, g, u \not\models \mathsf{ann}[\phi, b] = \mathsf{ann}[\forall x\phi_1, b] = \forall x\mathsf{ann}[\phi_1, b]$. There is $d \in \mathcal{D}$ such that $\mathfrak{M}, h[x \mapsto d], g, u \not\models \mathsf{ann}[\phi_1, b]$. By the contrapositive of the inductive hypothesis, $\mathfrak{M}, h[x \mapsto d], g, u \not\models \mathsf{ann}[\phi_1, a]$. Contradiction.
- $\phi = \exists x\phi_1$. Fix a, b with $a \le b$, $\mathfrak{M}$ with domain $\mathcal{D}$, and h, g, u. Suppose that $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi, a] = \mathsf{ann}[\exists x\phi_1, a] = \exists x\mathsf{ann}[\phi_1, a]$, which means that there is $d \in \mathcal{D}$ such that $\mathfrak{M}, h[x \mapsto d], g, u \models \mathsf{ann}[\phi_1, a]$. From the inductive hypothesis, we get that $\mathfrak{M}, h[x \mapsto d], g, u \models \mathsf{ann}[\phi_1, b]$ and therefore $\mathfrak{M}, h, g, u \models \exists x\mathsf{ann}[\phi_1, b] = \mathsf{ann}[\exists x\phi_1, b] = \mathsf{ann}[\phi, b]$.
- The cases $\phi = \bigcirc\phi_1, \newmoon\phi_1, \ominus\phi_1$ are easy.
- $\phi = \mu X\phi_1$. Fix a, b with $a \le b$ and $\mathfrak{M}, h, g, u$. Suppose that $\mathfrak{M}, h, g, u \models \mathsf{ann}[\mu X\phi_1, a] = \mu_\alpha X\mathsf{ann}[\phi_1, a]$, where $\alpha = a(X)$. We define the functions

$$f_1(S) = \{v \in \mathbb{N} \mid \mathfrak{M}, h, g[X \mapsto S], v \models \mathsf{ann}[\phi_1, a]\} = \|\mathsf{ann}[\phi_1, a]\|^{\mathfrak{M},h}_{g[X\mapsto S]}$$
$$f_2(S) = \{v \in \mathbb{N} \mid \mathfrak{M}, h, g[X \mapsto S], v \models \mathsf{ann}[\phi_1, b]\} = \|\mathsf{ann}[\phi_1, b]\|^{\mathfrak{M},h}_{g[X\mapsto S]}$$

  From the inductive hypothesis we get that for any $S \subseteq \mathbb{N}$, $f_1(S) \subseteq f_2(S)$. Let $\beta = b(X)$ and notice that $\alpha \le \beta$. It follows that $u \in \mu_\alpha(f_1) \subseteq \mu_\beta(f_1) \subseteq \mu_\beta(f_2)$. Therefore, $\mathfrak{M}, h, g, u \models \mu_\beta X\mathsf{ann}[\phi_1, b] = \mathsf{ann}[\mu X\phi_1, b]$.
- $\phi = \nu X\phi_1$. Fix a, b with $a \le b$ and h, g, u. Suppose that $\mathfrak{M}, h, g, u \models \mathsf{ann}[\nu X\phi_1, a] = \nu X\mathsf{ann}[\phi_1, a]$. We define the functions

$$f_1(S) = \|\mathsf{ann}[\phi_1, a]\|^{\mathfrak{M},h}_{g[X\mapsto S]} \qquad f_2(S) = \|\mathsf{ann}[\phi_1, b]\|^{\mathfrak{M},h}_{g[X\mapsto S]} \ .$$

  From the inductive hypothesis we get that for any $S \subseteq \mathbb{N}$, $f_1(S) \subseteq f_2(S)$. Therefore, $u \in \mathit{gfp}(f_1) \subseteq \mathit{gfp}(f_2)$ and hence $\mathfrak{M}, h, g, u \models \nu X\mathsf{ann}[\phi_1, b] = \mathsf{ann}[\nu X\phi_1, b]$. ☐

Corollary 22 (increasing ν-annotation)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-formula. Then, for any ν-annotations a, b for ϕ with $a \le b$, any first-order temporal structure $\mathfrak{M}$, any individual variable assignment h, any fixpoint variable assignment g, and any moment $u \in \mathbb{N}$,

$$\mathfrak{M}, h, g, u \not\models \mathsf{ann}[\phi, a] \implies \mathfrak{M}, h, g, u \not\models \mathsf{ann}[\phi, b] \ .$$

Proof. Let $\psi = \mathsf{pos}[\neg\phi]$, which means that $\phi = \mathsf{pos}[\neg\psi]$. Fix ν-annotations a, b for ϕ with $a \le b$, $\mathfrak{M}, h, g, u$. Notice that a, b are μ-annotations for ψ. Suppose that $\mathfrak{M}, h, g, u \not\models \mathsf{ann}[\phi, a]$, which implies that

$$\mathfrak{M}, h, g, u \models \mathsf{pos}[\neg\mathsf{ann}[\phi, a]] = \mathsf{pos}[\mathsf{ann}[\neg\phi, a]] \overset{\text{Lemma 19}}{=} \mathsf{ann}[\mathsf{pos}[\neg\phi], a] = \mathsf{ann}[\psi, a] \ .$$

From Lemma 21, we get that $\mathfrak{M}, h, g, u \models \mathsf{ann}[\psi, b]$ and hence

$$\mathfrak{M}, h, g, u \not\models \mathsf{pos}[\neg\mathsf{ann}[\psi, b]] = \mathsf{pos}[\mathsf{ann}[\neg\psi, b]] \overset{\text{Lemma 19}}{=} \mathsf{ann}[\mathsf{pos}[\neg\psi], b] = \mathsf{ann}[\phi, b] \ . \qquad ☐$$

Lemma 23. Fix a first-order temporal structure $\mathfrak{M}$. Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-formula. The following hold:

(1) For any individual variable assignment h, any fixpoint variable assignment g, and any moment $u \in \mathbb{N}$, if there is a μ-annotation a for ϕ such that $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi, a]$, then $\mathfrak{M}, h, g, u \models \phi$.

(2) There is a μ-annotation a for ϕ such that for any individual variable assignment h, any fixpoint variable assignment g, and any moment $u \in \mathbb{N}$,

$$\mathfrak{M}, h, g, u \models \phi \iff \mathfrak{M}, h, g, u \models \mathsf{ann}[\phi, a] \ .$$

Proof. Fix $\mathfrak{M}$ with domain $\mathcal{D}$. The proof proceeds by induction on ϕ.

- The cases $\phi = p, \neg p, X, \neg X, P(t_1, \dots, t_n), \neg P(t_1, \dots, t_n)$ are trivial.
- $\phi = (\phi_1 \wedge \phi_2)$. (1) Fix h, g, u. Suppose that there is a μ-annotation a for ϕ such that

$$\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi, a] = \mathsf{ann}[\phi_1, a] \wedge \mathsf{ann}[\phi_2, a] \ .$$

  Then, $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi_1, a]$ and $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi_2, a]$. From the inductive hypothesis (1), we have that $\mathfrak{M}, h, g, u \models \phi_1$ and $\mathfrak{M}, h, g, u \models \phi_2$, which implies that $\mathfrak{M}, h, g, u \models \phi_1 \wedge \phi_2 = \phi$.

(2) From the inductive hypothesis (2), we have that there is a μ-annotation a_i for ϕ_i such that for any h, g, $\|\phi_i\|_g^h = \|\mathsf{ann}[\phi, a_i]\|_g^h$. It is clear that $\mu\text{-}\mathsf{vars}[\phi] = \mu\text{-}\mathsf{vars}[\phi_1] \cup \mu\text{-}\mathsf{vars}[\phi_2]$. We define a μ-annotation a for ϕ as follows.

$$a(X) = \begin{cases} a_1(X), & \text{if } a_1(X) \text{ is defined \& } a_2(X) \text{ is undefined} \\ a_2(X), & \text{if } a_1(X) \text{ is undefined \& } a_2(X) \text{ is defined} \\ \sup\{a_1(X), a_2(X)\}, & \text{if both } a_1(X) \text{ and } a_2(X) \text{ are defined} \end{cases}$$

We show that a satisfies the property for ϕ. Fix h, g, u. (a) Observe that $a_1 \leq a$ and $a_2 \leq a$. Suppose that $\mathfrak{M}, h, g, u \models \phi = \phi_1 \wedge \phi_2$, which implies that $\mathfrak{M}, h, g, u \models \phi_1$ and $\mathfrak{M}, h, g, u \models \phi_2$. It follows that $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi_1, a_1]$ and $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi_2, a_2]$. From Lemma 21, we get that $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi_1, a]$ and $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi_2, a]$. We deduce that

$$\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi_1, a] \wedge \mathsf{ann}[\phi_2, a] = \mathsf{ann}[\phi_1 \wedge \phi_2, a] = \mathsf{ann}[\phi, a] \ .$$

(b) For the converse, suppose that

$$\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi, a] = \mathsf{ann}[\phi_1 \wedge \phi_2, a] = \mathsf{ann}[\phi_1, a] \wedge \mathsf{ann}[\phi_2, a] \ .$$

So, $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi_1, a]$ and $\mathfrak{M}, h, g, u \models \mathsf{ann}[\phi_2, a]$. From the inductive hypothesis (1), we get that $\mathfrak{M}, h, g, u \models \phi_1$ and $\mathfrak{M}, h, g, u \models \phi_2$. Hence $\mathfrak{M}, h, g, u \models \phi_1 \wedge \phi_2 = \phi$.

- The case $\phi = (\phi_1 \vee \phi_2)$ is similar to the previous one.
- $\phi = \forall x \phi_1$. (1) Fix h, g, u and suppose that there is a μ-annotation a for ϕ such that $\mathfrak{M}, h, g, u \models \mathsf{ann}[\forall x \phi_1, a] = \forall x \mathsf{ann}[\phi_1, a]$. Assume to the contrary that $\mathfrak{M}, h, g, u \not\models \forall x \phi_1$. There is $d \in \mathcal{D}$ such that $\mathfrak{M}, h[x \mapsto d], g, u \not\models \phi_1$. From the contrapositive of inductive hypothesis (1), we deduce that $\mathfrak{M}, h[x \mapsto d], g, u \not\models \mathsf{ann}[\phi_1, a]$. Contradiction. (2) From the inductive hypothesis (2), we get that there is a μ-annotation a for ϕ_1 such that

$$\|\phi_1\|_g^{\mathfrak{M},h} = \|\mathsf{ann}[\phi_1, a]\|_g^{\mathfrak{M},h} \ .$$

We argue that a satisfies the property for ϕ. Fix h, g, u. (a) Suppose that $\mathfrak{M}, h, g, u \models \forall x \phi_1$ and assume to the contrary that $\mathfrak{M}, h, g, u \not\models \mathsf{ann}[\forall x \phi_1, a] = \forall x \mathsf{ann}[\phi_1, a]$. There is $d \in \mathcal{D}$ such that $\mathfrak{M}, h, g, u \not\models \mathsf{ann}[\phi_1, a]$. From the inductive hypothesis (2), we get that $\mathfrak{M}, h, g, u \not\models \phi_1$. Contradiction. (b) Suppose that $\mathfrak{M}, h, g, u \models \mathsf{ann}[\forall x \phi_1, a] = \forall x \mathsf{ann}[\phi_1, a]$ and that, to the contrary, $\mathfrak{M}, h, g, u \not\models \forall x \phi_1$. There is $d \in \mathcal{D}$ such that $\mathfrak{M}, h[x \mapsto d], g, u \not\models \phi_1$. From the contrapositive of the inductive hypothesis (1), we get that $\mathfrak{M}, h[x \mapsto d], g, u \not\models \mathsf{ann}[\phi_1, a]$. Contradiction.
- The case $\phi = \exists x \phi_1$ is as straightforward as the previous one.
- The cases $\phi = \bigcirc\phi_1, \bullet\phi_1, \ominus\phi_1$ are easy.
- $\phi = \mu X \phi_1$. (1) Fix h, g, u and assume that there is a μ-annotation a for ϕ such that

$$\mathfrak{M}, h, g, u \models \mathsf{ann}[\mu X \phi_1, a] = \mu_\alpha X \mathsf{ann}[\phi_1, a], \text{ where } \alpha = a(X).$$

We define the functions

$$f_1(S) = \|\mathsf{ann}[\phi_1, a]\|_{g[X \mapsto S]}^{\mathfrak{M},h} \qquad\qquad f_2(S) = \|\phi_1\|_{g[X \mapsto S]}^{\mathfrak{M},h} \ .$$

From the inductive hypothesis (1), we get that for any $S \subseteq \mathbb{N}$, $f_1(S) \subseteq f_2(S)$. It follows that $u \in \mu_\alpha(f_1) \subseteq \mathit{lfp}(f_1) \subseteq \mathit{lfp}(f_2)$ and hence $\mathfrak{M}, h, g, u \models \mu X \phi_1$. (2) It follows from the inductive hypothesis (2) that there is a μ-annotation a_1 for ϕ_1 such that for any h, g,

$$\|\phi_1\|_g^{\mathfrak{M},h} = \|\mathsf{ann}[\phi_1, a_1]\|_g^{\mathfrak{M},h} \ .$$

Fix h, g, u and suppose that $\mathfrak{M}, h, g, u \models \mu X \phi_1$. There is an ordinal $\alpha = \alpha(h, g, u)$ such that $\mathfrak{M}, h, g, u \models \mu_\alpha X \phi_1$. The collection of all these ordinals is a set and since **Ord** is a proper class, there is an ordinal β that is greater than all these $\alpha(h, g, u)$. So, for any h, g, we get that

$$\|\mu X \phi_1\|_g^{\mathfrak{M},h} = \|\mu_\beta X \phi_1\|_g^{\mathfrak{M},h} \ .$$

Observe that $\mu\text{-}\mathsf{vars}[\mu X\phi_1] = \{X\} \cup \mu\text{-}\mathsf{vars}[\phi_1]$ and that $X \notin \mu\text{-}\mathsf{vars}[\phi_1]$. We will show that the μ-annotation a for ϕ, defined as

$$a(Y) = \begin{cases} \beta, & \text{if } Y = X \\ a_1(Y), & \text{if } Y \neq X \text{ and } Y \in \mu\text{-}\mathsf{vars}[\phi_1] \end{cases}$$

satisfies the property. First, we define the families of functions $\{f_1^{h,g}\}_{h,g}$, $\{f_2^{h,g}\}_{h,g}$ as

$$f_1^{h,g}(S) = \|\phi_1\|^{\mathfrak{M},h}_{g[X\mapsto S]} \qquad\qquad f_2^{h,g}(S) = \|\mathsf{ann}[\phi_1, a_1]\|^{\mathfrak{M},h}_{g[X\mapsto S]}$$

and observe that for any h, g and any $S \subseteq \mathbb{N}$, $f_1^{h,g}(S) = f_2^{h,g}(S)$. Fix h, g, u. We have that

$$\begin{aligned} \mathfrak{M}, h, g, u \models \mu X\phi_1 \iff \mathfrak{M}, h, g, u \models \mu_\beta X\phi_1 \iff u \in \mu_\beta(f_1^{h,g}) = \mu_\beta(f_2^{h,g}) \\ \iff \mathfrak{M}, h, g, u \models \mu_\beta X\mathsf{ann}[\phi_1, a_1] = \mu_\beta X\mathsf{ann}[\phi_1, a] = \mathsf{ann}[\mu X\phi_1, a] \ . \end{aligned}$$

- $\phi = \nu X\phi_1$. (1) Fix h, g, u. Suppose that there is a μ-annotation a for ϕ such that $\mathfrak{M}, h, g, u \models \mathsf{ann}[\nu X\phi_1, a] = \nu X\mathsf{ann}[\phi_1, a]$. We define the following functions.

$$f_1(S) = \|\mathsf{ann}[\phi_1, a]\|^{\mathfrak{M},h}_{g[X\mapsto S]} \qquad\qquad f_2(S) = \|\phi_1\|^{\mathfrak{M},h}_{g[X\mapsto S]}$$

It follows from the inductive hypothesis (1) that for any $S \subseteq \mathbb{N}$, $f_1(S) \subseteq f_2(S)$. Therefore, $u \in \mathit{gfp}(f_1) \subseteq \mathit{gfp}(f_2)$ and hence $\mathfrak{M}, h, g, u \models \nu X\phi_1$. (2) It follows from the inductive hypothesis (2) that there is a μ-annotation a_1 for ϕ_1 such that for any h, g,

$$\|\phi_1\|^{\mathfrak{M},h}_g = \|\mathsf{ann}[\phi_1, a_1]\|^{\mathfrak{M},h}_g \ .$$

We define the families of functions $\{f_1^{h,g}\}_{h,g}$ and $\{f_2^{h,g}\}_{h,g}$ as previously and observe that for any h, g and any $S \subseteq \mathbb{N}$, $f_1^{h,g}(S) = f_2^{h,g}(S)$. We argue that a_1 satisfies the property. Fix h, g, u. We have that

$$\begin{aligned} \mathfrak{M}, h, g, u \models \nu X\phi_1 \iff u \in \mathit{gfp}(f_1^{h,g}) = \mathit{gfp}(f_2^{h,g}) \\ \iff \mathfrak{M}, h, g, u \models \nu X\mathsf{ann}[\phi_1, a_1] = \mathsf{ann}[\nu X\phi_1, a_1] \ . \quad \Box \end{aligned}$$

Corollary 24. Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-formula. Then, for any first-order temporal structure $\mathfrak{M}$, any individual variable assignment h, any fixpoint variable assignment g, and any moment u,

$$\mathfrak{M}, h, g, u \models \phi \iff \text{there is a } \mu\text{-annotation } a \text{ for } \phi \text{ s.t. } \mathfrak{M}, h, g, u \models \mathsf{ann}[\phi, a].$$

Proof. It is an immediate consequence of Lemma 23. ☐

Corollary 25. Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-formula. Then, for any first-order temporal structure $\mathfrak{M}$, any individual variable assignment h, any fixpoint variable assignment g, and any moment u,

$$\mathfrak{M}, h, g, u \not\models \phi \iff \text{there is a } \nu\text{-annotation } a \text{ for } \phi \text{ s.t. } \mathfrak{M}, h, g, u \not\models \mathsf{ann}[\phi, a].$$

Proof. Fix $\phi, \mathfrak{M}, h, g, u$. Suppose that $\mathfrak{M}, h, g, u \not\models \phi$, which implies that $\mathfrak{M}, h, g, u \models \mathsf{pos}[\neg\phi]$. From Corollary 24 it follows that there is a μ-annotation a for $\mathsf{pos}[\neg\phi]$ such that

$$\mathfrak{M}, h, g, u \models \mathsf{ann}[\mathsf{pos}[\neg\phi], a] \overset{\text{Lemma 19}}{=} \mathsf{pos}[\mathsf{ann}[\neg\phi, a]] = \mathsf{pos}[\neg\mathsf{ann}[\phi, a]] \ .$$

Therefore, $\mathfrak{M}, h, g, u \not\models \mathsf{ann}[\phi, a]$. Observe that a is a ν-annotation for ϕ. ☐

4.3. **Results on Signatures.** We have already seen how formula satisfiability can be reduced to fp-sentence satisfiability. So, from now on, we will be limiting our attention to fp-sentences.

Consider a fixpoint fp-sentence ${}^{\mu}_{\nu}X\psi$. Observe that when unfolding to $[\psi]\{{}^{\mu}_{\nu}X\psi/X\}$, we do not run the risk of a free fixpoint variable in ${}^{\mu}_{\nu}X\psi$ getting bound. However, we still have to require that no free individual variable in ${}^{\mu}_{\nu}X\psi$ gets bound in $[\psi]\{{}^{\mu}_{\nu}X\psi/X\}$ if we want semantics to be preserved.

Definition 26 (fixpoint height)**.** Let ϕ be a $\mathcal{FOTL}_{\mu\nu}$-formula. We define the *fixpoint height* of ϕ to be the depth of nesting of fixpoint subformulas of ϕ. More formally we define $|\cdot|_{\mu\nu} : \mathcal{FOTL}_{\mu\nu} \to \mathbb{N}$ inductively as follows.

$$\begin{aligned}
|p|_{\mu\nu} &= 0 \\
|X|_{\mu\nu} &= 0 \\
|P(t_1,\dots,t_n)|_{\mu\nu} &= 0 \\
|\circ\phi|_{\mu\nu} &= |\phi|_{\mu\nu}, \text{where } \circ \in \{\neg, \bigcirc, \bullet, \ominus\} \\
|\phi_1 \otimes \phi_2|_{\mu\nu} &= \sup\{|\phi_1|_{\mu\nu}, |\phi_2|_{\mu\nu}\}\ , \text{ where } \otimes \in \{\wedge, \vee\} \\
|Qx\phi|_{\mu\nu} &= |\phi|_{\mu\nu}\ , \text{ where } Q \in \{\forall, \exists\} \\
|fX\phi|_{\mu\nu} &= 1 + |\phi|_{\mu\nu}\ , \text{ where } f \in \{\mu, \nu\}
\end{aligned}$$

For example, the fixpoint height of $\mu X(Px \vee \bigcirc X)$ is 1 and the fixpoint height of

$$\mu X[\nu Y(\neg Px \wedge \ominus^{10} Y) \vee (\exists y Rxy \wedge X)]$$

is 2.

We have already discussed that unfolding a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-formula $fX\phi$ into $[\phi]\{fX\phi/X\}$ does not necessarily yield a formula in positive normal form. We will develop a systematic way of renaming bound fixpoint variables of ϕ, thus getting $\dddot{\phi}$, so that $\dddot{\phi}$ is equal to ϕ modulo some bound fixpoint variables and $[\dddot{\phi}]\{fX\phi/X\}$ is in positive normal form. We have an uncountably infinite set of fixpoint variables $\mathcal{X} = \{X_0, X_1, X_2, X_3, \dots\}$ that are used in the construction of formulas. We introduce a infinite sequence of sets of fixpoint variables

$$\mathcal{X}' = \{X_0', X_1', X_2', \dots\} \qquad \mathcal{X}'' = \{X_0'', X_1'', X_2'', \dots\} \qquad \mathcal{X}''' = \{X_0''', X_1''', X_2''', \dots\} \qquad \dots$$

Think of these sets that they form layers as in Figure 1. The idea is that when we want to rename a fixpoint variable X we use the variable X' from the layer below. This way, we only introduce fresh variables. Moreover, if X, Y are in different columns, then X', Y' are in different columns and hence they are distinct. Observe that we only have to rename the fixpoint variables for fixpoint subformulas of ϕ in which X appears free. This is illustrated by the following example.

$$\begin{aligned}
\phi_1 = \mu X\psi_1 &= \mu X\{\nu Y(\neg p_1 \wedge \ominus^{10} Y) \vee [p_3 \wedge \bigcirc \mu Z(X \vee (p_2 \wedge \bigcirc Z))]\} \\
\dddot{\psi}_1 &= \underbrace{\nu Y(\neg p_1 \wedge \ominus^{10} Y)}_{\text{no renaming}} \vee [p_3 \wedge \bigcirc \underbrace{\mu Z'(X \vee (p_2 \wedge \bigcirc Z'))}_{\text{renaming}}] \\
[\dddot{\psi}_1]\{\phi_1/X\} &= \nu Y(\neg p_1 \wedge \ominus^{10} Y) \vee [p_3 \wedge \bigcirc \mu Z'(\phi_1 \vee (p_2 \wedge \bigcirc Z'))]
\end{aligned}$$

Let us work out a slightly more complex example.

$$\phi_1 = \mu X\psi_1 = \mu X \mu Y[\mu U(p \vee \bullet U) \vee \nu V(p \wedge \bigcirc X \wedge V) \vee \nu W(p \wedge X \wedge \bigcirc Y \wedge \ominus W)]$$

$$[\dddot{\psi}_1]\{\phi_1/X\} = \phi_2 = \mu Y'\psi_2 = \overbrace{\mu Y'[\underbrace{\mu U(p \vee \bullet U)}_{\text{no renaming}} \vee \underbrace{\nu V'(p \wedge \bigcirc\phi_1 \wedge V')}_{\text{renaming}} \vee \underbrace{\nu W'(p \wedge \phi_1 \wedge \bigcirc Y' \wedge \ominus W')}_{\text{renaming}}]}^{\text{renaming}}$$

$$\phi_3 = [\dddot{\psi}_2]\{\phi_2/Y'\} = \underbrace{\mu U(p \vee \bullet U)}_{\text{no renaming}} \vee \underbrace{\nu V'(p \wedge \bigcirc\phi_1 \wedge V')}_{\text{no renaming}} \vee \underbrace{\nu W''(p \wedge \phi_1 \wedge \bigcirc\phi_2 \wedge \ominus W'')}_{\text{renaming}}$$

Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-fp-sentence and X a fixpoint variable in ϕ. Clearly, X gets bound by some subformula $fX\psi$ of ϕ. We associate to X the fixpoint height of $fX\psi$. For the formulas ϕ_1, ϕ_2, ϕ_3 of the previous example we have

$$\begin{aligned}
A &= \{U \mapsto 1, V \mapsto 1, W \mapsto 1, Y \mapsto 2, X \mapsto 3\} \\
B &= A \cup \{V' \mapsto 4, W' \mapsto 4, Y' \mapsto 5\} \\
C &= B \cup \{W'' \mapsto 6\}
\end{aligned}$$

respectively. Observe that B agrees with A at the variables that ϕ_1, ϕ_2 share and that C agrees with B at the variables that ϕ_2, ϕ_3 share.

$$\begin{array}{ccccccc}
X_0 & X_1 & X_2 & X_3 & X_4 & X_5 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
X'_0 & X'_1 & X'_2 & X'_3 & X'_4 & X'_5 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
X''_0 & X''_1 & X''_2 & X''_3 & X''_4 & X''_5 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
X'''_0 & X'''_1 & X'''_2 & X'''_3 & X'''_4 & X'''_5 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{array}$$

FIGURE 1. Fixpoint variables used to ensure that all sentences in the closure of a $\mathcal{PTL}^{\mathsf{pnf}}_{\mu\nu}$-sentence are in positive normal form.

Definition 27 (positive closure, closure)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-fp-sentence. We define the *positive closure* of ϕ to be the smallest set $\mathsf{cl}^+[\phi]$ of fp-sentences that satisfies the following conditions:

(1) ϕ is in $\mathsf{cl}^+[\phi]$.
(2) If $(\psi_1 \otimes \psi_2) \in \mathsf{cl}^+[\phi]$, then $\psi_1, \psi_2 \in \mathsf{cl}^+[\phi]$, for $\otimes \in \{\wedge, \vee\}$.
(3) If $Qx\psi \in \mathsf{cl}^+[\phi]$, then $\psi \in \mathsf{cl}^+[\phi]$, for $Q \in \{\forall, \exists\}$.
(4) If $\circ\psi \in \mathsf{cl}^+[\phi]$, then $\psi \in \mathsf{cl}^+[\phi]$, for $\circ \in \{\bigcirc, \bullet, \ominus\}$.
(5) If $fX\psi \in \mathsf{cl}^+[\phi]$, then $[\dddot{\psi}]\{fX\psi/X\} \in \mathsf{cl}^+[\phi]$, for $f \in \{\mu, \nu\}$.

We define the *closure* of ϕ to be $\mathsf{cl}[\phi] = \mathsf{cl}^+[\phi] \cup \mathsf{cl}^+[\mathsf{neg}[\phi]]$. See Table 2 for an example. It is easy to see that $\mathsf{cl}^+[\mathsf{neg}[\phi]] = \mathsf{neg}[\mathsf{cl}^+[\phi]]$. From this we can deduce immediately that $\mathsf{cl}[\phi]$ is closed under $\mathsf{neg}[\cdot]$.

It should be noted that the closure of a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-fp-sentence contains only fp-sentences in positive normal form. We also define

$$V_\mu(\phi) = \bigcup_{\psi \in \mathsf{cl}[\phi]} \mu\text{-}\mathsf{vars}[\psi] \quad V_\nu(\phi) = \bigcup_{\psi \in \mathsf{cl}[\phi]} \nu\text{-}\mathsf{vars}[\psi] \quad V(\phi) = V_\mu(\phi) \cup V_\nu(\phi) = \bigcup_{\psi \in \mathsf{cl}[\phi]} \mu\nu\text{-}\mathsf{vars}[\psi] \ .$$

To each fp-sentence ψ in $\mathsf{cl}[\phi]$ we associate a function $d_\psi : \mu\nu\text{-}\mathsf{vars}[\psi] \to \mathbb{N}$, which maps each fixpoint variable in $\mu\nu\text{-}\mathsf{vars}[\psi]$ to the fixpoint depth of the fixpoint subformula of ψ that binds it. It is easy to see that d_ψ, d'_ψ agree on the fixpoint variables in $\mu\nu\text{-}\mathsf{vars}[\psi] \cap \mu\nu\text{-}\mathsf{vars}[\psi']$. It follows that $d = \bigcup_{\psi \in \mathsf{cl}[\phi]} d_\psi$ is a function from $V(\phi)$ to $\mathbb{N}$. Obviously, d induces a strict partial order on $V(\phi)$

$$X < Y \overset{\text{def}}{\iff} d(X) < d(Y) \ ,$$

which can be easily extended to some strict linear order $<$. See Table 3.

Lemma 28. The cardinality of $\mathsf{cl}[\phi]$ is linear in the length of ϕ.

Proof. Easy. □

Definition 29 (μ_ϕ-annotation, ν_ϕ-annotation, $<_\phi$)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-fp-sentence. A *$\mu_\phi$-annotation* a for $\psi \in \mathsf{cl}[\phi]$ is a μ-annotation for ψ that is defined exactly on $V_\mu(\phi)$. Similarly, a *ν_ϕ-annotation* a for $\psi \in \mathsf{cl}[\phi]$ is a ν-annotation for ψ that is defined exactly on $V_\nu(\phi)$.

The reason we introduce μ_ϕ-annotations and ν_ϕ-annotation is that they can be easily well-ordered lexicographically as described in Definition 17. The order in which we consider the fixpoint variables in $V_\mu(\phi)$ and $V_\nu(\phi)$ is important. For μ_ϕ-annotations a, b, we will write $a <_\phi b$ to mean $a <_{V_\mu(\phi)} b$. For ν_ϕ-annotations a, b, we will write $a <_\phi b$ to mean $a <_{V_\nu(\phi)} b$. We also define

$$a \leq_\phi b \overset{\text{def}}{\iff} a <_\phi b \text{ or } a = b \ .$$

Definition 30 (μ_ϕ-signature, ν_ϕ-signature)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-fp-sentence.

TABLE 2. Closure of a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-fp-sentence.

$\phi = \ominus\forall x\neg Rxz \wedge \exists y\mu X(\bigcirc Py \vee \bullet^3 X)$ $\quad\tilde{\phi} = \mathsf{neg}[\phi] = \bullet\exists xRxz \vee \forall y\nu X(\bigcirc\neg Py \wedge \ominus^3 X)$

	$\mathsf{cl}^+[\phi]$	$\mathsf{cl}^+[\tilde{\phi}]$
1	$\ominus\forall x\neg Rxz \wedge \exists y\mu X(\bigcirc Py \vee \bullet^3 X)$	$\bullet\exists xRxz \vee \forall y\nu X(\bigcirc\neg Py \wedge \ominus^3 X)$
2	$\ominus\forall x\neg Rxz$	$\bullet\exists xRxz$
3	$\forall x\neg Rxz$	$\exists xRxz$
4	$\neg Rxz$	Rxz
5	$\exists y\mu X(\bigcirc Py \vee \bullet^3 X)$	$\forall y\nu X(\bigcirc\neg Py \wedge \ominus^3 X)$
6	$\phi_1 = \mu X(\bigcirc Py \vee \bullet^3 X)$	$\tilde{\phi}_1 = \nu X(\bigcirc\neg Py \wedge \ominus^3 X)$
7	$\bigcirc Py \vee \bullet^3\phi_1$	$\bigcirc\neg Py \wedge \ominus^3\tilde{\phi}_1$
8	$\bigcirc Py$	$\bigcirc\neg Py$
9	Py	$\neg Py$
10	$\bullet^3\phi_1$	$\ominus^3\tilde{\phi}_1$
11	$\bullet^2\phi_1$	$\ominus^2\tilde{\phi}_1$
12	$\bullet\phi_1$	$\ominus\tilde{\phi}_1$

TABLE 3. Closure of a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-fo-sentence and ordering of fixpoint variables.

$$\phi_1 = \mu X\big\{\nu Y(\neg Py \wedge \ominus^{10} Y) \vee [Qx \wedge \bigcirc\mu Z(X \vee (Rxy \wedge \bigcirc Z))]\big\}$$
$$\phi_2 = \nu Y(\neg Py \wedge \ominus^{10} Y) \vee [Qx \wedge \bigcirc\mu Z'(\phi_1 \vee (Rxy \wedge \bigcirc Z'))]$$
$$\tilde{\phi}_1 = \mathsf{neg}[\phi_1] = \nu X\big\{\mu Y(Py \vee \bullet^{10} Y) \wedge [\neg Qx \vee \bigcirc\nu Z(X \wedge (\neg Rxy \vee \bigcirc Z))]\big\}$$
$$\tilde{\phi}_2 = \mathsf{neg}[\phi_2] = \mu Y(Py \vee \bullet^{10} Y) \wedge [\neg Qx \vee \bigcirc\nu Z'(\phi_1 \wedge (\neg Rxy \vee \bigcirc Z'))]$$

i	$\mathsf{cl}^+[\phi]$	$\mathsf{cl}^+[\tilde{\phi}]$	$d_i(X)$	$d_i(Y)$	$d_i(Z)$	$d_i(Z')$
1	ϕ_1	$\tilde{\phi}_1$	2	1	1	
2	ϕ_2	$\tilde{\phi}_2$	2	1	1	3
3	$\phi_3 = \nu Y(\neg Py \wedge \ominus^{10} Y)$	$\tilde{\phi}_3 = \mu Y(Py \vee \bullet^{10} Y)$		1		
4	$\neg Py \wedge \ominus^{10}\phi_3$	$Py \vee \bullet^{10}\tilde{\phi}_3$		1		
5	$\neg Py$	Py				
6	$\ominus^{10}\phi_3$	$\bullet^{10}\tilde{\phi}_3$		1		
7	$\ominus^{9}\phi_3$	$\bullet^{9}\tilde{\phi}_3$		1		
$\vdots$	$\vdots$	$\vdots$		$\vdots$		
15	$\ominus\phi_3$	$\bullet\tilde{\phi}_3$		1		
16	$Qx \wedge \bigcirc\mu Z'(\phi_1 \vee (Rxy \wedge \bigcirc Z'))$	$\neg Qx \vee \bigcirc\nu Z'(\tilde{\phi}_1 \wedge (\neg Rxy \vee \bigcirc Z'))$	2	1	1	3
17	Qx	$\neg Qx$				
18	$\bigcirc\mu Z'(\phi_1 \vee (Rxy \wedge \bigcirc Z'))$	$\bigcirc\nu Z'(\tilde{\phi}_1 \wedge (\neg Rxy \vee \bigcirc Z'))$	2	1	1	3
19	$\phi_{19} = \mu Z'(\phi_1 \vee (Rxy \wedge \bigcirc Z'))$	$\tilde{\phi}_{19} = \nu Z'(\phi_1 \wedge (\neg Rxy \vee \bigcirc Z'))$	2	1	1	3
20	$\phi_1 \vee (Rxy \wedge \bigcirc\phi_{19})$	$\tilde{\phi}_1 \wedge (\neg Rxy \vee \bigcirc\tilde{\phi}_{19})$	2	1	1	3
21	$Rxy \wedge \bigcirc\phi_{19}$	$\neg Rxy \vee \bigcirc\tilde{\phi}_{19}$	2	1	1	3
22	Rxy	$\neg Rxy$				
23	$\bigcirc\phi_{19}$	$\bigcirc\tilde{\phi}_{19}$	2	1	1	3

- We say that $\psi \in \mathsf{cl}[\phi]$ has *μ_ϕ-signature a at moment u under $\mathfrak{M}, h$* if a is the $<_\phi$-least μ_ϕ-annotation for ψ such that $\mathfrak{M}, h, u \models \mathsf{ann}[\psi, a]$.
- We say that $\psi \in \mathsf{cl}[\phi]$ has *ν_ϕ-signature a at moment u under $\mathfrak{M}, h$* if a is the $<_\phi$-least ν_ϕ-annotation for ψ such that $\mathfrak{M}, h, u \not\models \mathsf{ann}[\psi, a]$.

A fp-sentence $\psi \in \mathsf{cl}[\phi]$ has μ_ϕ-signature at u under $\mathfrak{M}, h$ if and only if $\mathfrak{M}, h, u \models \psi$ (Corollary 24). Similarly, $\psi \in \mathsf{cl}[\phi]$ has ν_ϕ-signature at u under $\mathfrak{M}, h$ if and only if $\mathfrak{M}, h, u \not\models \psi$ (Corollary 25).

Fix a fp-sentence $\psi \in \mathsf{cl}[\phi]$ with μ_ϕ-signature (ν_ϕ-signature) a at u under h and a fixpoint variable X in $V_\mu(\phi)$ ($V_\nu(\phi)$) that is not in μ-$\mathsf{vars}[\psi]$ (ν-$\mathsf{vars}[\psi]$). It is obvious that $a(X) = 0$.

Lemma 31. Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-fp-sentence and $\psi \in \mathsf{cl}[\phi]$. Then, for any first-order temporal structure $\mathfrak{M}$, any individual variable assignment h, and any moment $u \in \mathbb{N}$, ψ has μ_ϕ-signature a at u under $\mathfrak{M}, h$ if and only if $\mathsf{neg}[\psi]$ has ν_ϕ-signature a at u under $\mathfrak{M}, h$.

Proof. Let b be an arbitrary μ_ϕ-annotation for ψ. So, b is a ν_ϕ-annotation for $\mathsf{neg}[\psi]$. Fix a first-order temporal structure $\mathfrak{M}$, an individual variable assignment h, and a moment u. We have that

$$\mathfrak{M}, h, u \models \mathsf{ann}[\psi, b] \iff \mathfrak{M}, h, u \not\models \mathsf{pos}[\neg\mathsf{ann}[\psi, b]] = \mathsf{pos}[\mathsf{ann}[\neg\psi, b]] \overset{\text{Lemma 19}}{=} \mathsf{ann}[\mathsf{pos}[\neg\psi], b] \ .$$

Suppose that ψ has μ_ϕ-signature a_1 at u under $\mathfrak{M}, h$ and $\mathsf{neg}[\psi]$ has ν_ϕ-signature a_2 at u under $\mathfrak{M}, h$. From the $\Rightarrow$ direction, we have that $a_2 \leq_\phi a_1$ and from the $\Leftarrow$ direction that $a_1 \leq_\phi a_2$. It follows that $a_1 = a_2$. ☐

Proposition 32 (μ_ϕ-signatures)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-fp-sentence.

(1) If $(\psi_1 \wedge \psi_2)$ is in $\mathsf{cl}[\phi]$ and has μ_ϕ-signature a at moment u under $\mathfrak{M}, h$, then both ψ_1 and ψ_2 have μ_ϕ-signatures $\leq_\phi a$ at u under $\mathfrak{M}, h$.
(2) If $(\psi_1 \vee \psi_2)$ is in $\mathsf{cl}[\phi]$ and has μ_ϕ-signature a at moment u under $\mathfrak{M}, h$, then ψ_1 or ψ_2 has μ_ϕ-signature $\leq_\phi a$ at u under $\mathfrak{M}, h$.
(3) If $\forall x\psi$ is in $\mathsf{cl}[\phi]$ and has μ_ϕ-signature a at moment u under $\mathfrak{M}, h$, then for any d in $\mathsf{dom}(\mathfrak{M})$, ψ has μ_ϕ-signature $\leq_\phi a$ at u under $\mathfrak{M}, h[x \mapsto d]$.
(4) If $\exists x\psi$ is in $\mathsf{cl}[\phi]$ and has μ_ϕ-signature a at moment u under $\mathfrak{M}, h$, then there is d in $\mathsf{dom}(\mathfrak{M})$ such that ψ has μ_ϕ-signature $\leq_\phi a$ at u under $\mathfrak{M}, h[x \mapsto d]$.
(5) If $\bigcirc\psi$ is in $\mathsf{cl}[\phi]$ and has μ_ϕ-signature a at moment u under $\mathfrak{M}, h$, then ψ has μ_ϕ-signature a at $(u+1)$ under $\mathfrak{M}, h$.
(6) If $\bullet\psi$ is in $\mathsf{cl}[\phi]$ and has μ_ϕ-signature a at moment $u > 0$ under $\mathfrak{M}, h$, then ψ has μ_ϕ-signature a at $(u-1)$ under $\mathfrak{M}, h$.
(7) If $\ominus\psi$ is in $\mathsf{cl}[\phi]$ and has μ_ϕ-signature a at moment $u > 0$ under $\mathfrak{M}, h$, then ψ has μ_ϕ-signature a at $(u-1)$ under $\mathfrak{M}, h$.
(8) If $\mu X\psi$ is in $\mathsf{cl}[\phi]$, has μ_ϕ-signature a at moment u under $\mathfrak{M}, h$, and no free individual variable in $\mu X\psi$ gets bound in $[\dddot{\psi}]\{\mu X\psi/X\}$, then $a(X)$ is a successor ordinal and $[\dddot{\psi}]\{\mu X\psi/X\}$ has μ_ϕ-signature $\leq_\phi b$ at u under $\mathfrak{M}, h$, where $b(X) = a(X) - 1$, $b(Y) = a(Y)$ for any $Y \in \mu$-$\mathsf{vars}[\psi]$, and $b(Z') = b(Z)$ for any newly introduced $Z' \in \mu$-$\mathsf{vars}[\dddot{\psi}] \setminus \mu$-$\mathsf{vars}[\psi]$.
(9) If $\nu X\psi$ is in $\mathsf{cl}[\phi]$, has μ_ϕ-singnature a at moment u under $\mathfrak{M}, h$, and no free individual variable in $\nu X\psi$ gets bound in $[\dddot{\psi}]\{\nu X\psi/X\}$, then $[\dddot{\psi}]\{\nu X\psi/X\}$ has μ_ϕ-signature $\leq_\phi b$ at u under $\mathfrak{M}, h$, where $b(Y) = a(Y)$ for any $Y \in \mu$-$\mathsf{vars}[\psi]$, and $b(Z') = b(Z)$ for any newly introduced $Z' \in \mu$-$\mathsf{vars}[\dddot{\psi}] \setminus \mu$-$\mathsf{vars}[\psi]$.

Proof. Fix a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-fp-sentence ϕ.

(1) Suppose that $(\psi_1 \wedge \psi_2)$ is in $\mathsf{cl}[\phi]$ and has μ_ϕ-signature a at u under $\mathfrak{M}, h$. This means that $\mathfrak{M}, h, u \models \mathsf{ann}[\psi_1 \wedge \psi_2, a] = \mathsf{ann}[\psi_1, a] \wedge \mathsf{ann}[\psi_2, a]$, which implies that $\mathfrak{M}, h, u \models \mathsf{ann}[\psi_1, a]$ and $\mathfrak{M}, h, u \models \mathsf{ann}[\psi_2, a]$. It immediately follows that both ψ_1 and ψ_2 have μ_ϕ-signatures $\leq_\phi a$ at u under $\mathfrak{M}, h$.
(2) Similarly to (1).

(3) Suppose that $\forall x\psi$ is in $\mathsf{cl}[\phi]$ and has μ_ϕ-signature a at u under $\mathfrak{M}, h$. This means that $\mathfrak{M}, h, u \models \mathsf{ann}[\forall x\psi, a] = \forall x \mathsf{ann}[\psi, a]$. Let d be an arbitrary element of $\mathsf{dom}(\mathfrak{M})$. We have that $\mathfrak{M}, h[x \mapsto d], u \models \mathsf{ann}[\psi, a]$ and hence ψ has μ_ϕ-signature $\leq_\phi a$ at u under $\mathfrak{M}, h[x \mapsto d]$.
(4) The case for $\exists x\psi$ is similar to the previous one.
(5) Let b be an arbitrary μ_ϕ-annotation. For any $\mathfrak{M}, h, u$, it holds that

$$\mathfrak{M}, h, u \models \mathsf{ann}[\bigcirc\psi, b] = \bigcirc\mathsf{ann}[\psi, b] \iff \mathfrak{M}, h, u+1 \models \mathsf{ann}[\psi, b] \ .$$

Suppose that $\bigcirc\psi$ is in $\mathsf{cl}[\phi]$ and has μ_ϕ-signature a_1 at u under $\mathfrak{M}, h$. Let a_2 be the μ_ϕ-signature of ψ at $(u+1)$ under $\mathfrak{M}, h$. From the $\Rightarrow$ direction we get that $a_2 \leq_\phi a_1$ and from the $\Leftarrow$ direction that $a_1 \leq_\phi a_2$. It follows that $a_1 = a_2$.
(6) Similarly to (5).
(7) Similarly to (6).
(8) Suppose that $\mu X\psi$ is in $\mathsf{cl}[\phi]$, has μ_ϕ-signature a at u under $\mathfrak{M}, h$, and no free individual variable in $\mu X\psi$ gets bound in $[\dddot{\psi}]\{\mu X\psi/X\}$. First, notice that $a(X)$ is a successor ordinal. Suppose for contradiction that $a(X)$ is a limit ordinal λ. We have that $\mathfrak{M}, h, u \models \mathsf{ann}[\mu X\psi, a] = \mu_\lambda X \mathsf{ann}[\psi, a]$, which means that

$$u \in \mu_\lambda(f) = \bigcup_{\alpha<\lambda} \mu_\alpha(f) \ , \text{ where } f(S) = \|\mathsf{ann}[\psi, a]\|^{\mathfrak{M},h}_{X\mapsto S} \ .$$

So, there is an ordinal $\alpha < \lambda$ such that $u \in \mu_\alpha(f)$ and hence $\mathfrak{M}, h, u \models \mu_\alpha X \mathsf{ann}[\psi, a]$. Consider the μ_ϕ-annotation $\hat{a} = a[X \mapsto \alpha]$ for ψ. We have that $\mathsf{ann}[\psi, \hat{a}] = \mathsf{ann}[\psi, a]$ and therefore $\mathfrak{M}, h, u \models \mu_\alpha X \mathsf{ann}[\psi, \hat{a}] = \mathsf{ann}[\mu X\psi, \hat{a}]$. But, $\hat{a} < a$ and hence $\hat{a} <_\phi a$, which is a contradiction. We have established that $a(X)$ is a successor ordinal $(\alpha+1)$.

We define a μ_ϕ-annotation b as $b(X) = \alpha = a(X) - 1$, $b(Y) = a(Y)$ for any $Y \in \mu\text{-}\mathsf{vars}[\psi]$ and $b(Z') = b(Z)$ for any newly introduced $Z' \in \mu\text{-}\mathsf{vars}[\dddot{\psi}] \setminus \mu\text{-}\mathsf{vars}[\psi]$. Observe that a, b agree on $\mu\text{-}\mathsf{vars}[\psi]$ and that $\mathsf{ann}[\psi, b]$, $\mathsf{ann}[\dddot{\psi}, b]$ are equal modulo the names of some bound fixpoint variables. Define $f'(S) = \|\mathsf{ann}[\psi, b]\|^{\mathfrak{M},h}_{X\mapsto S}$. We have that

$$\begin{aligned}
&\mathfrak{M}, h, u \models \mathsf{ann}[\mu X\psi, a] = \mu_{\alpha+1} X \mathsf{ann}[\psi, a] = \mu_{\alpha+1} X \mathsf{ann}[\psi, b] \implies \\
&\qquad u \in \mu_{\alpha+1}(f') = f'(\mu_\alpha(f')) \implies \mathfrak{M}, h, X \mapsto \mu_\alpha(f'), u \models \mathsf{ann}[\psi, b] \implies \\
&\qquad\qquad \mathfrak{M}, h, X \mapsto \mu_\alpha(f'), u \models \mathsf{ann}[\dddot{\psi}, b] \overset{\text{Lemma 8}}{\implies} \\
&\mathfrak{M}, h, u \models [\mathsf{ann}[\dddot{\psi}, b]]\{\mu_\alpha X \mathsf{ann}[\psi, b]/X\} = [\mathsf{ann}[\dddot{\psi}, b]]\{\mathsf{ann}[\mu X\psi, b]/X\} \\
&\qquad\qquad\qquad \overset{\text{Lemma 18}}{=} \mathsf{ann}[[\dddot{\psi}]\{\mu X\psi/X\}, b] \ .
\end{aligned}$$

It follows that $[\dddot{\psi}]\{\mu X\psi/X\}$ has μ_ϕ-signature $\leq_\phi b$ at u under $\mathfrak{M}, h$.
(9) Suppose that $\nu X\psi$ is in $\mathsf{cl}[\phi]$, has μ_ϕ-signature a at u under $\mathfrak{M}, h$, and no free individual variable in $\nu X\psi$ gets bound in $[\dddot{\psi}]\{\nu X\psi/X\}$. We define a μ_ϕ-annotation b as $b(Y) = a(Y)$ for any $Y \in \mu\text{-}\mathsf{vars}[\psi]$ and $b(Z') = b(Z)$ for any newly introduced $Z' \in \mu\text{-}\mathsf{vars}[\dddot{\psi}] \setminus \mu\text{-}\mathsf{vars}[\psi]$. Observe that $\mathsf{ann}[\psi, b]$, $\mathsf{ann}[\dddot{\psi}, b]$ are equal modulo the names of some bound fixpoint variables. Define $f(S) = \|\mathsf{ann}[\psi, b]\|^{\mathfrak{M},h}_{X\mapsto S}$. We have that

$$\begin{aligned}
&\mathfrak{M}, h, u \models \mathsf{ann}[\nu X\psi, a] = \nu X \mathsf{ann}[\psi, a] = \nu X \mathsf{ann}[\psi, b] \implies \\
&\qquad u \in \mathit{gfp}(f) = f(\mathit{gfp}(f)) \implies \mathfrak{M}, h, X \mapsto \mathit{gfp}(f), u \models \mathsf{ann}[\psi, b] \implies \\
&\qquad\qquad \mathfrak{M}, h, X \mapsto \mathit{gfp}(f), u \models \mathsf{ann}[\dddot{\psi}, b] \overset{\text{Lemma 8}}{\implies} \\
&\mathfrak{M}, h, u \models [\mathsf{ann}[\dddot{\psi}, b]]\{\nu X \mathsf{ann}[\psi, b]/X\} = [\mathsf{ann}[\dddot{\psi}, b]]\{\mathsf{ann}[\nu X\psi, b]/X\} \\
&\qquad\qquad\qquad \overset{\text{Lemma 18}}{=} \mathsf{ann}[[\dddot{\psi}]\{\nu X\psi/X\}, b] \ .
\end{aligned}$$

It follows that $[\dddot{\psi}]\{\nu X\psi/X\}$ has μ_ϕ-signature $\leq_\phi b$ at u under $\mathfrak{M}, h$. □

Proposition 33 (ν_ϕ-signatures)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-fp-sentence.

(1) If $(\psi_1 \wedge \psi_2)$ is in $\mathsf{cl}[\phi]$ and has ν_ϕ-signature a at moment u under $\mathfrak{M}, h$, then ψ_1 or ψ_2 has ν_ϕ-signature $\leq_\phi a$ at u under $\mathfrak{M}, h$.

(2) If $(\psi_1 \vee \psi_2)$ is in $\mathsf{cl}[\phi]$ and has ν_ϕ-signature a at moment u under $\mathfrak{M}, h$, then both ψ_1 and ψ_2 have ν_ϕ-signatures $\leq_\phi a$ at u under $\mathfrak{M}, h$.
(3) If $\forall x \psi$ is in $\mathsf{cl}[\phi]$ and has ν_ϕ-signature a at moment u under $\mathfrak{M}, h$, then there is d in $\mathsf{dom}(\mathfrak{M})$ such that ψ has ν_ϕ-signature $\leq_\phi a$ at u under $\mathfrak{M}, h[x \mapsto d]$.
(4) If $\exists x \psi$ is in $\mathsf{cl}[\phi]$ and has ν_ϕ-signature a at moment u under $\mathfrak{M}, h$, then for any d in $\mathsf{dom}(\mathfrak{M})$, ψ has ν_ϕ-signature $\leq_\phi a$ at u under $\mathfrak{M}, h[x \mapsto d]$.
(5) If $\bigcirc \psi$ is in $\mathsf{cl}[\phi]$ and has ν_ϕ-signature a at moment u under $\mathfrak{M}, h$, then ψ has ν_ϕ-signature a at $(u+1)$ under $\mathfrak{M}, h$.
(6) If $\bullet \psi$ is in $\mathsf{cl}[\phi]$ and has ν_ϕ-signature a at moment $u > 0$ under $\mathfrak{M}, h$, then ψ has ν_ϕ-signature a at $(u-1)$ under $\mathfrak{M}, h$.
(7) If $\ominus \psi$ is in $\mathsf{cl}[\phi]$ and has ν_ϕ-signature a at moment $u > 0$ under $\mathfrak{M}, h$, then ψ has ν_ϕ-signature a at $(u-1)$ under $\mathfrak{M}, h$.
(8) If $\mu X \psi$ is in $\mathsf{cl}[\phi]$, has ν_ϕ-singnature a at moment u under $\mathfrak{M}, h$, and no free individual variable in $\mu X \psi$ gets bound in $[\dddot{\psi}]\{\mu X \psi / X\}$, then $[\dddot{\psi}]\{\mu X \psi / X\}$ has ν_ϕ-signature $\leq_\phi b$ at u under $\mathfrak{M}, h$, where $b(Y) = a(Y)$ for any $Y \in \nu\text{-}\mathsf{vars}[\psi]$, and $b(Z') = b(Z)$ for any newly introduced $Z' \in \nu\text{-}\mathsf{vars}[\dddot{\psi}] \setminus \nu\text{-}\mathsf{vars}[\psi]$.
(9) If $\nu X \psi$ is in $\mathsf{cl}[\phi]$, has ν_ϕ-signature a at moment u under $\mathfrak{M}, h$, and no free individual variable in $\nu X \psi$ gets bound in $[\dddot{\psi}]\{\nu X \psi / X\}$, then $a(X)$ is a successor ordinal and $[\dddot{\psi}]\{\nu X \psi / X\}$ has ν_ϕ-signature $\leq_\phi b$ at u under $\mathfrak{M}, h$, where $b(X) = a(X) - 1$, $b(Y) = a(Y)$ for any $Y \in \nu\text{-}\mathsf{vars}[\psi]$, and $b(Z') = b(Z)$ for any newly introduced $Z' \in \nu\text{-}\mathsf{vars}[\dddot{\psi}] \setminus \nu\text{-}\mathsf{vars}[\psi]$.

Proof. It is an immediate consequence of Proposition 32 and Lemma 31. Let us show (8). Fix a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mu\nu}$-fp-sentence ϕ and $\mu X \psi \in \mathsf{cl}[\phi]$. Suppose that $\mu X \psi$ has ν_ϕ-signature a (at u under $\mathfrak{M}, h$) and that no free individual variable in $\mu X \psi$ gets bound in $[\dddot{\psi}]\{\mu X \psi / X\}$. From Lemma 31, we get that $\sim\mu X \psi = \nu X {\sim}[\psi]\{\neg X / X\}$ has μ_ϕ-signature a. Let $\psi' = {\sim}[\psi]\{\neg X / X\}$ and observe that $\dddot{\psi}' = {\sim}[\dddot{\psi}]\{\neg X / X\}$. By Proposition 32, $[{\sim}[\dddot{\psi}]\{\neg X / X\}]\{\sim\mu X \psi / X\} = {\sim}[\dddot{\psi}]\{\mu X \psi / X\}$ (Lemma 13) has μ_ϕ-signature $\leq_\phi b$, where b is defined as in (8). From Lemma 31, we deduce that $[\dddot{\psi}]\{\mu X \psi / X\}$ has ν_ϕ-signature $\leq_\phi b$. $\square$

5. The Monodic Fragment

The monodic fragment of first-order temporal logic (with no fixpoint operators) is defined in [32] as the subset of first-order temporal formulas ϕ such that any subformula of ϕ beginning with a temporal operator has at most one free individual variable. If the pure first-order part is decidable, then the monodic fragment is known to be decidable over several classes of flows of time [32].

Naturally, when temporal fixpoint operators are added, a similar monodicity restriction will have to be applied to μ/ν-subformulas, i.e. 'any μ/ν-subformula has at most one free individual variable'. Without this restriction, we immediately deduce undecidability. It is clear that the F ('sometime in the future') connective can be expressed with a least fixpoint as

$$\mathsf{F}\phi \equiv \bigcirc\phi \vee \bigcirc\mathsf{F}\phi \equiv \mu X[\bigcirc\phi \vee \bigcirc X] \ .$$

So, if we allow two free individual variables in μ-subformulas, we can express $\mathsf{F}\phi(x, y)$ and encode the $\mathbb{N} \times \mathbb{N}$ recurrent tiling problem (see Theorem 2 in [32]), which is Σ^1_1-complete [25].

Another restriction will also be enforced, namely that a $\forall/\exists$-subformula does not contain any free fixpoint variables. This requirement, even though is appears a bit different from the monodicity restriction of [32], serves a similar purpose. The idea of monodicity is that we move through time "on a single domain element". This is why, for example, a formula of the form $\bigcirc\phi$ is not allowed to have two free individual variables. But, during the evaluation of fixpoints we move through time. If along the way of the evaluation we come across a quantifier (this happens only when the fixpoint variable of the fixpoint being evaluated is free below the quantifier), we "move" to another domain element.

Definition 34 (monodic $\mathcal{FOTL}_{\mu\nu}$-formulas)**.** We define the set of $\mathcal{FOTL}_{\mathbf{1}\mu\nu}$-formulas to be the subset of $\mathcal{FOTL}_{\mu\nu}$ that contains a $\mathcal{FOTL}_{\mu\nu}$-formula ϕ iff

- any subformula of ϕ beginning with a temporal or fixpoint operator has at most one free individual variable and
- any subformula of ϕ beginning with a quantifier has no free fixpoint variables.

That is, $\phi \in \mathcal{FOTL}_{\mathbf{1}\mu\nu}$ iff the following hold.

(1) $\phi \in \mathcal{FOTL}_{\mu\nu}$
(2) For any $\circ \in \{\bigcirc, \bullet, \ominus\}$ and any $\circ\psi \in \mathsf{sbf}[\phi]$, $|\mathsf{fvars}[\circ\psi]| \leq 1$.
(3) For any $f \in \{\mu, \nu\}$ and any $fX\psi \in \mathsf{sbf}[\phi]$, $|\mathsf{fvars}[fX\psi]| \leq 1$.
(4) For any $Q \in \{\forall, \exists\}$ and any $Qx\psi \in \mathsf{sbf}[\phi]$, $\mathsf{fp\text{-}free}[Qx\psi] = \emptyset$.

$\mathcal{FOTL}^{\mathsf{pos}}_{\mathbf{1}\mu\nu}$ is the set of monodic $\mathcal{FOTL}_{\mu\nu}$-formulas in positive form and $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}$ is the set of monodic $\mathcal{FOTL}_{\mu\nu}$-formulas in positive normal form.

For example, the formula $\exists x \mu X(\bigcirc\forall y Rxy \vee \bullet^2 X)$ is monodic, whereas the formula $\exists x\forall y\mu X(Rxy \vee \bullet^3 X)$ is not.

Observe that we can always unfold a fixpoint monodic $\mathcal{FOTL}_{\mathbf{1}\mu\nu}$-fp-sentence with the semantics being preserved. Let $fX\phi$ be a $\mathcal{FOTL}_{\mathbf{1}\mu\nu}$-fp-sentence. As far as free fixpoint variables are concerned, we are done: $fX\phi$ has no free fixpoint variables. Assume now that $fX\phi$ has a free individual variable, namely x. Does it get bound in $[\phi]\{fX\phi/X\}$? For contradiction, assume that it gets bound. This means that there is a subformula of $Qx\psi$ of ϕ, where Q is a quantifier, such that X occurs free in ψ. This contradicts the monodic restriction.

We will denote by $\mathcal{FOTL}^k_{\mu\nu}$ the set of $\mathcal{FOTL}_{\mu\nu}$-formulas that have at most k free individual variables.

6. Quasimodels

It may be helpful to view the quasimodel technique of [32] as a (much more involved) generalization of the filtration technique used for the propositional case. In order to decide satisfiability of a formula ϕ of propositional temporal logic (without fixpoint operators), each moment is associated — in an object called pre-model — with a maximal Boolean-consistent subset of the subformulas of ϕ (type). The pre-model also satisfies obvious temporal constraints. The idea is then to show that the pre-model records at each moment exactly those subformulas of ϕ that are true at the particular moment under the propositional variable assignment that the pre-model defines. Hence, ϕ is satisfied in a temporal structure if and only if it is satisfied in a pre-model.

In the first-order case (without fixpoints), the situation is considerably more complicated. The quasimodel technique can also handle constants, but we will ignore them for now. The monodicity restriction allows us to consider only those subformulas of ϕ that have at most one free variable. Call them sub_1. A type is then a maximal Boolean-consistent subset of sub_1. Under a given first-order temporal structure, truth for a formula of sub_1 is considered with respect to a specific moment and a specific domain element. Consider, at a given moment, the elements of the domain that satisfy the same formulas of sub_1. We group them under the type they satisfy. So, we define a quasimodel that records at each moment a *set of types*, which can be thought to partition the elements of the domain. Suppose that we fix a domain element and assign the free variable of each formula of sub_1 to this domain element. We are then down to the propositional case and certain temporal consistency restrictions are satisfied. This is encoded by special functions, called runs, that correspond to domain elements. The crucial question is whether this information is sufficient to reconstruct a model out of it. Towards this end, we want the set of types attached to each moment to represent a first-order structure. This property will be called realizability. In order to check this, we have to make the formulas purely first-order, by replacing the maximal subformulas that begin with a temporal operator with a unary predicate symbol or propositional variable called surrogate. We will not go into further details at this point. Suffice it to say that what the quasimodel does is separate in a way the first-order part from the temporal part. The algorithm that checks realizability is plugged-in. The temporal part is handled largely by the runs.

The addition of fixpoint operators in the case of propositional temporal logic, creates the need to introduce the concept of well-foundedness. The idea is very roughly that least fixpoints need a finite number of steps to be evaluated, whereas for greatest fixpoints this is not the case. In order to "guide" the evaluation of least fixpoints, Streett and Emerson use "choice functions" that direct the evaluation towards some disjunct of a disjunction [53]. For the case of monodic first-order temporal logic with fixpoint operators the same technique is applied. It is not a very difficult extension, since at the level of a run things are in a way "propositional".

6.1. The notion of well-founded quasimodel.

We will continue to define types, surrogates, state candidates, realizable state candidates, state functions, and quasimodels. In order to handle the fixpoints, we also introduce choice functions, adorned quasimodels and define the central notion of well-foundedness of a quasimodel.

Definition 35 ($\mathsf{cl}_1^+, \mathsf{cl}_1, \mathsf{cl}_0$)**.** Let ϕ be a $\mathcal{FOTL}_{\mu\nu}[\sigma]$-fp-sentence. We define $\mathsf{cl}_1^+[\phi]$ to be the subset of $\mathsf{cl}^+[\phi]$ that contains only the fp-sentences that have at most one free individual variable. Similarly, we define $\mathsf{cl}_1[\phi]$ to be the subset of $\mathsf{cl}[\phi]$ that contains only the fp-sentences that have at most one free individual variable. $\mathsf{cl}_0[\phi]$ is the subset of $\mathsf{cl}[\phi]$ that contains only sentences.

Definition 36 (type)**.** A *type* for a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence ϕ is a subset t of $\mathsf{cl}_1[\phi]$ that satisfies the following conditions.

1. For any $\psi \in \mathsf{cl}_1[\phi]$, $\psi \in t \iff \mathsf{neg}[\phi] \notin t$.
2. For any $(\psi_1 \wedge \psi_2) \in \mathsf{cl}_1[\phi]$, $(\psi_1 \wedge \psi_2) \in t \iff \psi_1 \in t$ and $\psi_2 \in t$.
3. For any $(\psi_1 \vee \psi_2) \in \mathsf{cl}_1[\phi]$, $(\psi_1 \vee \psi_2) \in t \iff \psi_1 \in t$ or $\psi_2 \in t$.
4. For any $\mu X \psi \in \mathsf{cl}_1[\phi]$, $\mu X \psi \in t \iff [\psi]\{\mu X\psi/X\} \in t$.
5. For any $\nu X \psi \in \mathsf{cl}_1[\phi]$, $\nu X \psi \in t \iff [\psi]\{\nu X\psi/X\} \in t$.

It follows from the first condition that a type for ϕ contains exactly $\left|\mathsf{cl}_1^+[\phi]\right| = \left|\mathsf{cl}_1^+[\mathsf{neg}[\phi]]\right|$ elements. We denote by $\mathsf{Types}(\phi)$ the set of all types for ϕ. Clearly, $\mathsf{Types}(\phi)$ is a subset of the powerset of $\mathsf{cl}_1[\phi]$. It follows that there are at most

$$\flat(\phi) = |\wp(\mathsf{cl}_1[\phi])| = 2^{|\mathsf{cl}_1[\phi]|}$$

different types for ϕ.

Definition 37 (renaming the free individual variable)**.** We define the function

$$[\cdot]_\star : \mathcal{FOTL}^1_{\mu\nu} \times \left(\mathcal{V} \cup \mathcal{C} \cup \{\_\}\right) \to \mathcal{FOTL}^1_{\mu\nu} \quad [\phi]_x = \begin{cases} \phi, & \text{if } \phi \text{ has no free variables} \\ [\phi]\{x/y\} & \text{if } \phi \text{ has } y \text{ as its unique free variable} \end{cases}$$

That is, the function renames the free individual variable in ϕ if one exists, or leaves ϕ unchanged if it is a dom-sentence. The symbol $\_$ acts as a "placeholder" for the free individual variable. It is obvious that properties like

$$[\phi_1 \wedge \phi_2]_x = [\phi_1]_x \wedge [\phi_2]_x$$

hold, because of the inductive definition of substitution.

Definition 38 (surrogates)**.** For every $\mathcal{FOTL}\mathbf{1}_{\mu\nu}[\sigma]$-formula ψ that begins with a temporal or fixpoint operator and has one free individual variable we introduce a unary predicate symbol $P_{[\psi]_\_}$. For every $\mathcal{FOTL}\mathbf{1}_{\mu\nu}[\sigma]$-dom-sentence χ that begins with a temporal or fixpoint operator we introduce a propositional variable p_χ. $P_{[\psi]_\_}$ and p_χ are called the *surrogates* of ψ and χ respectively. For example,

$$\psi = \bigcirc(Pu \wedge \forall v Quv) \qquad\qquad P_{[\psi]_\_} = P_{\bigcirc(P\_ \wedge \forall v Q\_v)}$$
$$\chi = \mu X(\exists x \forall y Rxy \vee \bullet^2 X) \qquad\qquad p_\chi = p_{\mu X(\exists x \forall y Rxy \vee \bullet^2 X)}$$

Given the signature

$$\sigma = (\mathcal{P}, \mathcal{P}_0, \mathcal{F} = \emptyset, \mathcal{C}, \mathsf{ar})$$

for our language, we fix a new signature

$$\sigma_{\mathsf{surr}} = (\mathcal{P}', \mathcal{P}'_0, \mathcal{F} = \emptyset, \mathcal{C}, \mathsf{ar}')$$
$$\mathcal{P}' = \mathcal{P} \cup \left\{P_{[\psi]_\_} \mid \psi \text{ is a } (\bigcirc/\bullet/\ominus/\mu/\nu)\text{-}\mathcal{FOTL}\mathbf{1}_{\mu\nu}[\sigma]\text{-formula with } |\mathsf{fvars}[\psi]| = 1\right\}$$
$$\mathcal{P}'_0 = \mathcal{P}_0 \cup \left\{p_\chi \mid \chi \text{ is a } (\bigcirc/\bullet/\ominus/\mu/\nu)\text{-}\mathcal{FOTL}\mathbf{1}_{\mu\nu}[\sigma]\text{-dom-sentence}\right\}$$
$$\mathsf{ar}'[P] = \begin{cases} \mathsf{ar}[P], & \text{if } P \in \mathcal{P}; \\ 1, & \text{if } P \text{ is a surrogate.} \end{cases}$$

that contains the surrogates.

Definition 39 (reduct)**.** Let ϕ be a $\mathcal{FOTL}\mathbf{1}_{\mu\nu}[\sigma]$-formula. We define the function

$$\overline{\cdot} : \mathcal{FOTL}\mathbf{1}_{\mu\nu}[\sigma] \to \mathcal{FOL}[\sigma_{\mathsf{surr}}] \ ,$$

where $\mathcal{FOL}[\sigma_{\mathsf{surr}}]$ is the first-order language (no temporal or fixpoint operators) over the signature σ_{surr}. $\overline{\phi}$ is the first-order formula that results by replacing its maximal temporal or fixpoint subformulas by their surrogates. $\overline{\phi}$ is called the *reduct* of ϕ. More formally,

$$\overline{p} = p \qquad \overline{\neg\phi} = \neg\overline{\phi} \qquad \overline{\forall x\phi} = \forall x\overline{\phi}$$
$$\overline{P(t_1, \ldots, t_n)} = P(t_1, \ldots, t_n) \qquad \overline{\phi_1 \wedge \phi_2} = \overline{\phi_1} \wedge \overline{\phi_2} \qquad \overline{\exists x\phi} = \exists x\overline{\phi}$$
$$\overline{X} = X \qquad \overline{\phi_1 \vee \phi_2} = \overline{\phi_1} \vee \overline{\phi_2}$$

$$\overline{\circ\phi} = \begin{cases} p_{\circ\phi}, & \text{if } \circ\phi \text{ is a sentence;} \\ P_{[\circ\phi]_\_}(x), & \text{if } x \text{ is the unique free variable in } \circ\phi \end{cases}, \text{ for all } \circ \in \{\bigcirc, \bullet, \ominus\}$$

$$\overline{fX\phi} = \begin{cases} p_{fX\phi}, & \text{if } fX\phi \text{ is a sentence;} \\ P_{[fX\phi]_\_}(x), & \text{if } x \text{ is the unique free variable in } fX\phi \end{cases}, \text{ for all } f \in \{\mu, \nu\}$$

Definition 40 (state candidate)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence, T be a set of types for ϕ and T^{con} a function from $\mathsf{con}[\phi]$ to T. Then, the pair (T, T^{con}) is called a *state candidate* for ϕ.

We denote by $\mathsf{StateC}(\phi)$ the set of all state candidates for ϕ. The number of distinct state candidates for ϕ is at most

$$\sharp(\phi) = 2^{\flat(\phi)} \cdot \flat(\phi)^{|\mathsf{con}[\phi]|} \ .$$

Since T is a set of types for $\phi \in \mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$, $T \in \wp\left(\wp\left(\mathsf{cl}_1[\phi]\right)\right)$, it can be constructed at most in

$$|\wp\left(\wp\left(\mathsf{cl}_1[\phi]\right)\right)| = 2^{|\wp(\mathsf{cl}_1[\phi])|} = 2^{\flat(\phi)}$$

ways. T^{con} contains exactly $|\mathsf{con}[\phi]|$ ordered pairs. The type in each one of these pairs can be chosen among at most $\flat(\phi)$ as it has already been shown. Therefore, T^{con} can be constructed at most in

$$\underbrace{\flat(\phi) \times \flat(\phi) \times \flat(\phi) \times \cdots \times \flat(\phi)}_{|\mathsf{con}[\phi]| \text{ times}} = \flat(\phi)^{|\mathsf{con}[\phi]|}$$

ways. The value for $\sharp(\phi)$ follows immediately.

Definition 41. Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence, $\mathfrak{D} = (\mathcal{D}, \cdot^{\mathfrak{D}})$ be a first-order structure over the signature σ_{surr}, and $a \in \mathcal{D}$. We define the set

$$t^{\mathfrak{D}}(a) = \left\{\psi \in \mathsf{cl}_1[\phi] \mid \mathfrak{D}, z \mapsto a \models [\overline{\psi}]_z\right\} \ .$$

Assume that z is a reserved individual variable that never occurs in a formula. This way we can be certain that z does not get bound in $[\overline{\psi}]_z$. $t^{\mathfrak{D}}(a)$ is the set of formulas in $\mathsf{cl}_1[\phi]$ that are true in the structure $\mathfrak{D}$ under any variable assignment that maps their free variable (if any) to a. It is easy to see that $t^{\mathfrak{D}}(a)$ is a type for ϕ.

Definition 42 (realizable state candidate)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence. We say that the first-order structure $\mathfrak{D}$ over σ_{surr} *realizes* a state candidate (T, T^{con}) for ϕ iff the following conditions are true.

$$T = \left\{t^{\mathfrak{D}}(a) \mid a \in \mathcal{D}\right\} \qquad\qquad T^{con}(c) = t^{\mathfrak{D}}(c^{\mathfrak{D}}) \text{ , for all } c \in \mathsf{con}[\phi]$$

A state candidate is said to be *finitely realizable* iff there exists a finite first-order structure that realizes it. A *realizable state candidate* is a state candidate that is realized by some first-order structure $\mathfrak{D}$.

We denote by $\mathsf{Real}(\phi)$ the set of all realizable state candidates for ϕ. Clearly, $\mathsf{Real}(\phi)$ is a subset of $\mathsf{StateC}(\phi)$ and hence there can be at most $\sharp(\phi)$ distinct realizable state candidates.

Remark 43. Let $\mathfrak{D} = (\mathcal{D}, \cdot^{\mathfrak{D}})$ be a first-order structure over the signature σ_{surr} and (T, T^{con}) be a state candidate that is realized by $\mathfrak{D}$. Then, T defines a partitioning of the domain $\mathcal{D}$. We define the relation $\sim_{\mathfrak{D}} \subseteq \mathcal{D} \times \mathcal{D}$:

$$\text{for any } a, b \in \mathcal{D},\ a \sim_{\mathfrak{D}} b \iff t^{\mathfrak{D}}(a) = t^{\mathfrak{D}}(b) \ .$$

$\sim_{\mathfrak{D}}$ is obviously reflexive, symmetric, and transitive. Hence, it is an equivalence relation.

We say that two types t and t' for a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence ϕ *agree* on $\mathsf{cl}_0[\phi]$ iff $t \cap \mathsf{cl}_0[\phi] = t' \cap \mathsf{cl}_0[\phi]$, i.e. they contain the same sentences.

Lemma 44 (types of realizable state candidates agree on $\mathsf{cl}_0[\phi]$)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence and (T, T^{con}) a realizable state candidate for ϕ. Then, all types in T agree on $\mathsf{cl}_0[\phi]$, i.e. for any $t, t' \in T$, $t \cap \mathsf{cl}_0[\phi] = t' \cap \mathsf{cl}_0[\phi]$.

Proof. Since (T, T^{con}) is a realizable state candidate, there exists a first-order structure $\mathfrak{D} = (\mathcal{D}, \cdot^{\mathfrak{D}})$ over the signature σ_{surr} such that

$$T = \left\{t^{\mathfrak{D}}(a) \mid a \in \mathcal{D}\right\} \qquad\qquad t^{\mathfrak{D}}(a) = \left\{\psi \in \mathsf{cl}_1[\phi] \mid \mathfrak{D}, z \mapsto a \models [\overline{\psi}]_z\right\} \ .$$

Assume to the contrary that there exist $t, t' \in T$ such that $t \cap \mathsf{cl}_0[\phi] \neq t' \cap \mathsf{cl}_0[\phi]$. Without loss of generality we assume that there exists ψ such that $\psi \in t \cap \mathsf{cl}_0[\phi]$ and $\psi \notin t' \cap \mathsf{cl}_0[\phi]$. There are $a, b \in \mathcal{D}$ such that $t = t^{\mathfrak{D}}(a)$ and $t' = t^{\mathfrak{D}}(b)$. We have that $\psi \in \mathsf{cl}_0[\phi]$, which means that ψ

is a sentence. Therefore, its truth value its independent of the variable assignment. It follows that

$$\mathfrak{D}, z \mapsto a \models [\overline{\psi}]_z = \overline{\psi} \iff \mathfrak{D} \models \overline{\psi} \iff \mathfrak{D}, z \mapsto b \models \overline{\psi} = [\overline{\psi}]_z \ .$$

We immediately deduce that $\psi \in t^{\mathfrak{D}}(b) = t'$ and therefore $\psi \in t' \cap \mathsf{sbf}^0_{\neg}[\phi]$. Contradiction. $\square$

Lemma 45. Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence, t a type for ϕ, $\mathfrak{D} = (\mathcal{D}, \cdot^{\mathfrak{D}})$ a first-order structure over σ_{surr}, and $a \in \mathcal{D}$. Then,

$$\mathfrak{D}, z \mapsto a \models \bigwedge_{\psi \in t} [\overline{\psi}]_z \quad \text{ if and only if } \quad t = t^{\mathfrak{D}}(a) \ .$$

Proof. The $\Leftarrow$ direction is trivial. Let us show the $\Rightarrow$ direction. • $t \subseteq t^{\mathfrak{D}}(a)$. Let $\psi \in t$. Since t is a type for ϕ, $\psi \in \mathsf{cl}_1[\phi]$. By satisfaction of the conjunction, we also get that $\mathfrak{D}, z \mapsto a \models [\overline{\psi}]_z$. Therefore, $\psi \in t^{\mathfrak{D}}(a)$. • $t^{\mathfrak{D}}(a) \subseteq t$. Let $\psi \in t^{\mathfrak{D}}(a)$. Then, $\psi \in \mathsf{cl}_1[\phi]$ and $\mathfrak{D}, z \mapsto a \models [\overline{\psi}]_z$. Assume for contradiction that $\psi \notin t$. It follows that $\mathsf{neg}[\psi] \in t$ and therefore

$$\mathfrak{D}, z \mapsto a \models [\overline{\mathsf{neg}[\psi]}]_z = [\mathsf{neg}[\overline{\psi}]]_z = \mathsf{neg}[[\overline{\psi}]_z] \ ,$$

which implies that $\mathfrak{D}, z \mapsto a \not\models [\overline{\psi}]_z$. Contradiction. $\square$

Lemma 46. Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence and $\mathfrak{C} = (T, T^{con})$ be a state candidate for ϕ. $\mathfrak{C}$ is (finitely) realizable if and only if the $\mathcal{FOL}[\sigma_{\mathsf{surr}}]$-sentence

$$\alpha_{\mathfrak{C}} = \left(\bigwedge_{t \in T} \exists z \bigwedge_{\psi \in t} [\overline{\psi}]_z \right) \wedge \left(\forall z \bigvee_{t \in T} \bigwedge_{\psi \in t} [\overline{\psi}]_z \right) \wedge \left(\bigwedge_{c \in \mathsf{con}[\phi]} \bigwedge_{\psi \in T^{con}(c)} [\overline{\psi}]_c \right)$$

is satisfied in some (finite) first-order structure.

Proof. Suppose that (T, T^{con}) is realizable. Then, there exists a first-order structure $\mathfrak{D} = (\mathcal{D}, \cdot^{\mathfrak{D}})$ over σ_{surr} such that

$$T = \left\{ t^{\mathfrak{D}}(a) \mid a \in \mathcal{D} \right\} \quad \text{and} \quad T^{con}(c) = t^{\mathfrak{D}}(c^{\mathfrak{D}}) \text{ , for all } c \in \mathsf{con}[\phi] \ .$$

We will prove that $\alpha_{\mathfrak{C}}$ is satisfied in $\mathfrak{D}$, i.e. $\mathfrak{D} \models \alpha_{\mathfrak{C}}$.

- Let t be an arbitrary type in T. Then, there exists $a \in \mathcal{D}$ such that $t = t^{\mathfrak{D}}(a)$. By definition of $t^{\mathfrak{D}}(a)$, $\mathfrak{D}, z \mapsto a \models [\overline{\psi}]_z$, for all $\psi \in t^{\mathfrak{D}}(a) = t$, which implies that $\mathfrak{D}, z \mapsto a \models \bigwedge_{\psi \in t} [\overline{\psi}]_z$. It follows that $\mathfrak{D} \models \exists z \bigwedge_{\psi \in t} [\overline{\psi}]_z$.
- Let a be an arbitrary element of $\mathcal{D}$. Then, $t = t^{\mathfrak{D}}(a) \in T$ and as before we can prove that $\mathfrak{D}, z \mapsto d \models \bigwedge_{\psi \in t} [\overline{\psi}]_z$.
- Let c be an arbitrary constant in $\mathsf{con}[\phi]$ and ψ an arbitrary formula in $T^{con}(c) = t^{\mathfrak{D}}(c^{\mathfrak{D}})$. By definition of $t^{\mathfrak{D}}(c^{\mathfrak{D}})$, $\mathfrak{D}, z \mapsto c^{\mathfrak{D}} \models [\overline{\psi}]_z$, which implies that $\mathfrak{D} \models [\overline{\psi}]_c$.

Suppose now that the sentence $\alpha_{\mathfrak{C}}$ is satisfied in a first-order structure $\mathfrak{D} = (\mathcal{D}, \cdot^{\mathfrak{D}})$, i.e. $\mathfrak{D} \models \alpha_{\mathfrak{C}}$. We will prove that $\mathfrak{C} = (T, T^{con})$ is realized by $\mathfrak{D}$, that is

$$T = \left\{ t^{\mathfrak{D}}(a) \mid a \in \mathcal{D} \right\} \quad \text{and} \quad T^{con}(c) = t^{\mathfrak{D}}(c^{\mathfrak{D}}) \text{ , for all } c \in \mathsf{con}[\phi] \ .$$

- $T \subseteq \{t^{\mathfrak{D}}(a) \mid a \in \mathcal{D}\}$. Let $t \in T$. By satisfaction of $\alpha_{\mathfrak{C}}$ (first conjunct), we get that $\mathfrak{D} \models \exists z \bigwedge_{\psi \in t} [\overline{\psi}]_z$, which means that there exists $a \in \mathcal{D}$ such that $\mathfrak{D}, z \mapsto a \models \bigwedge_{\psi \in t} [\overline{\psi}]_z$. By Lemma 45, $t = t^{\mathfrak{D}}(a)$.
- $\{t^{\mathfrak{D}}(a) \mid a \in \mathcal{D}\} \subseteq T$. Let us consider $t^{\mathfrak{D}}(a)$, where $a \in \mathcal{D}$. By satisfaction of the formula $\alpha_{\mathfrak{C}}$ (second conjunct), we get that there exists $t \in T$ such that $\mathfrak{D}, z \mapsto a \models \bigwedge_{\psi \in t} [\overline{\psi}]_z$. By Lemma 45, $t = t^{\mathfrak{D}}(a)$.
- Let c be an arbitrary constant in $\mathsf{con}[\phi]$. By satisfaction of $\alpha_{\mathfrak{C}}$ (third conjunct), we get that $\mathfrak{D} \models \bigwedge_{\psi \in T^{con}(c)} [\overline{\psi}]_c$, which implies that $\mathfrak{D}, z \mapsto c^{\mathfrak{D}} \models \bigwedge_{\psi \in T^{con}(c)} [\overline{\psi}]_z$. By Lemma 45, $T^{con}(c) = t^{\mathfrak{D}}(c^{\mathfrak{D}})$. $\square$

Definition 47 (state function)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence. A *state function* for ϕ is a function f that maps each $u \in \mathbb{N}$ to a realizable state candidate $f(u) = (T_u, T^{con}_u)$ for ϕ. We will often write $T_u = f(u).T$ and $T^{con}_u = f(u).T^{con}$.

Definition 48 (run)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence and f a state function for ϕ. A *run* r in f is a function in $\prod_{u \in \mathbb{N}} f(u).T$ such that the following hold.

- For every $\bigcirc\psi \in \mathsf{cl}_1[\phi]$ and every $u \in \mathbb{N}$, $\bigcirc\psi \in r(u) \iff \psi \in r(u+1)$.
- For every $\CIRCLE\psi \in \mathsf{cl}_1[\phi]$ and every $u \in \mathbb{N}$, $\CIRCLE\psi \in r(u) \iff u > 0$ and $\psi \in r(u-1)$.
- For every $\ominus\psi \in \mathsf{cl}_1[\phi]$ and every $u \in \mathbb{N}$, $\ominus\psi \in r(u) \iff u = 0$ or $[u > 0$ and $\psi \in r(u-1)]$.

Definition 49 (quasimodel, choice function, adorned quasimodel, finitary quasimodel)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}$-sentence, f be a state function for ϕ, and $\mathscr{R}$ be a set of runs in f. The pair $\mathfrak{m} = (f, \mathscr{R})$ is a *quasimodel* for ϕ if the following hold.

(1) For any $u \in \mathbb{N}$ and any $t \in f(u).T$, there is $r \in \mathscr{R}$ such that $r(u) = t$.
(2) For every $c \in \mathsf{con}[\phi]$, the function r_c, defined as $r_c(u) = f(u).T^{con}(c)$, is a run in $\mathscr{R}$.

We define the set of *occurrences* in a quasimodel $\mathfrak{m} = (f, \mathscr{R})$ as

$$Occ(\mathfrak{m}) = \{(\psi, r, u) \mid u \in \mathbb{N}, r \in \mathscr{R}, \psi \in r(u)\} .$$

A *choice function* for a quasimodel $\mathfrak{m}$ is a function that maps each occurrence $(\psi_1 \vee \psi_2, r, u)$ of a disjunction in $\mathfrak{m}$ to a disjunct ψ_i provided that ψ_i is in $r(u)$. An *adorned quasimodel* for ϕ is a pair $(\mathfrak{m}, \tau)$, where $\mathfrak{m}$ is a quasimodel for ϕ and τ is a choice function for $\mathfrak{m}$.

Since the choice function τ for a quasimodel $\mathfrak{m} = (f, \mathscr{R})$ takes a run as a parameter, we can also see it as a family of functions $\{\tau_r\}_{r \in \mathscr{R}}$, where $\tau_r(\psi_1 \vee \psi_2, u) = \tau(\psi_1, \psi_2, r, u)$. An *adorned run* in f is a pair (r, τ_r), where r is a run in f and τ_r is a *choice function* for r, i.e. a function that maps each disjunction in a type $r(u)$ to a disjunct that is in $r(u)$.

A quasimodel $\mathfrak{m} = (f, \mathscr{R})$ is called *finitary* if for every $u \in \mathbb{N}$, the quasistate $f(u)$ is finitely realizable and the set of runs $\mathscr{R}$ is finite.

Definition 50 (derivation on runs, regeneration, well-foundedness)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence and $(\mathfrak{m}, \tau)$ an adorned quasimodel for ϕ. We define the *derivation relation* $\vdash_{\mathfrak{m},\tau}$ on the occurrences $Occ(\mathfrak{m})$ as follows (we drop the parentheses around occurrences to increase readability).

(1) If $(\psi_1 \wedge \psi_2, r, u) \in Occ(\mathfrak{m})$, then $\psi_1 \wedge \psi_2, r, u \vdash_{\mathfrak{m},\tau} \psi_1, r, u$ and $\psi_1 \wedge \psi_2, r, u \vdash_{\mathfrak{m},\tau} \psi_2, r, u$.
(2) If $(\psi_1 \vee \psi_2, r, u) \in Occ(\mathfrak{m})$, then $\psi_1 \vee \psi_2, r, u \vdash_{\mathfrak{m},\tau} \tau(\psi_1 \vee \psi_2, r, u), r, u$.
(3) If $(\bigcirc\psi, r, u) \in Occ(\mathfrak{m})$, then $\bigcirc\psi, r, u \vdash_{\mathfrak{m},\tau} \psi, r, u+1$.
(4) If $(\CIRCLE\psi, r, u) \in Occ(\mathfrak{m})$, then $u > 0$ (by definition) and $\CIRCLE\psi, r, u \vdash_{\mathfrak{m},\tau} \psi, r, u-1$.
(5) If $(\ominus\psi, r, u) \in Occ(\mathfrak{m})$ and $u > 0$, then $\ominus\psi, r, u \vdash_{\mathfrak{m},\tau} \psi, r, u-1$.
(6) If $(\mu X\psi, r, u) \in Occ(\mathfrak{m})$, then $\mu X\psi, r, u \vdash_{\mathfrak{m},\tau} [\psi]\{\mu X\psi/X\}, r, u$.
(7) If $(\nu X\psi, r, u) \in Occ(\mathfrak{m})$, then $\nu X\psi, r, u \vdash_{\mathfrak{m},\tau} [\psi]\{\nu X\psi/X\}, r, u$.

The transitive closure of $\vdash_{\mathfrak{m},\tau}$ is denoted by $\vdash^+_{\mathfrak{m},\tau}$ and its reflexive transitive closure by $\vdash^*_{\mathfrak{m},\tau}$. It is easy to see that from a fp-sentence $\psi \in \mathsf{cl}_1[\phi]$ we can only derive fp-sentences in $\mathsf{cl}_1[\psi]$.

For every run $r \in \mathscr{R}$, and every least fixpoint fp-sentence $\mu X\psi_X$ in $\mathsf{cl}_1[\phi]$, we define a *regeneration relation* $R^{\mathfrak{m},\tau}_{r,X}$ on the set of moments $\{u \in \mathbb{N} \mid \mu X\psi_X \in r(u)\}$ as: $(u, v) \in R^{\mathfrak{m},\tau}_{r,X}$ iff there is a finite sequence of occurrences $(\psi_1, r, u_1), (\psi_2, r, u_2), \ldots, (\psi_k, r, u_k)$ such that

- $\psi_1 = \mu X\psi_X$ and $u_1 = u$,
- $\psi_k = \mu X\psi_X$ and $u_k = v$,
- $\psi_i, r, u_i \vdash_{\mathfrak{m},\tau} \psi_{i+1}, r, u_{i+1}$ for all $i \in \{1, \ldots, k-1\}$,
- $\mu X\psi_X$ is a subformula of ψ_i for all $i \in \{1, \ldots, k\}$.

We say that $(\mathfrak{m}, \tau)$ is *well-founded* if for every run $r \in \mathscr{R}$ and every least fixpoint sentence $\mu X\psi_X$ in $\mathsf{cl}_1[\phi]$, the regeneration relation $R^{\mathfrak{m},\tau}_{r,X}$ is converse well-founded.

Definition 51 (satisfaction in quasimodels)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence. We say that ϕ is *satisfied in a quasimodel* $\mathfrak{m} = (f, \mathscr{R})$ for ϕ if there is a moment $u \in \mathbb{N}$ such that ϕ is in all types in $f(u).T$. It is *satisfied in an adorned quasimodel* $(\mathfrak{m}, \tau)$ if it is satisfied in $\mathfrak{m}$.

6.2. **Satisfaction in model implies satisfaction in well-founded quasimodel.** Starting from a temporal structure, in which ϕ is satisfied, it is relatively straightforward to construct a quasimodel that satisfies ϕ. At each moment, the set of types of the state candidate is defined by taking all the types that are satisfied at some element under the temporal structure. Each constant is mapped to the type that is satisfied at the domain element that is the interpretation of the constant. Easily, we show that the state candidates are realized by a first-order structure that extends the local first-order structure of the temporal structure by interpreting the surrogates in the same way the corresponding temporal formulas are interpreted by the temporal structure. For each domain element, a run is defined that follows through time the types satisfied at this particular element. It remains to "adorn" the quasimodel with an appropriate choice function and prove its well-foundedness. The choice function selects the disjuncts with the least signatures. The converse well-foundedness of the regeneration relations is shown by noticing that the first least fixpoint unfolding strictly decreases the signature and that this decrease cannot be compensated for.

Theorem 52. Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence. If ϕ is satisfiable, then there is a well-founded adorned quasimodel for ϕ, in which ϕ is satisfied.

Proof. Fix a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence ϕ and suppose that ϕ is satisfied in some first-order temporal structure $\big(\langle\mathbb{N},<\rangle,\mathcal{D},I\big)$ over the signature σ at $w\in\mathbb{N}$. Let t^u_a denote the set of fp-sentences in $\mathsf{cl}_1[\phi]$ that are true in $\mathfrak{M}$ at the moment u and at the domain element a. That is, for all $u\in\mathbb{N}$ and for all $a\in\mathcal{D}$,

$$t^u_a=\big\{\psi\in\mathsf{cl}_1[\phi]\mid\mathfrak{M},z\mapsto a,u\models[\psi]_z\big\}\ .$$

It is clear that all t^u_a are types for ϕ. We also define

$$T_u=\big\{t^u_a\mid a\in\mathcal{D}\big\}\qquad\qquad T^{con}_u(c)=t^u_{c^{Iw}}\qquad\qquad f(u)=(T_u,T^{con}_u)$$

We claim that for all $u\in\mathbb{N}$, $f(u)$ is a realizable state candidate. Let u be an arbitrary moment in $\mathbb{N}$. In order to prove this claim we will construct a first-order structure $\mathfrak{N}_u=(\mathcal{D},\cdot^{\mathfrak{N}_u})$ over the signature σ_{surr} that realizes $f(u)$. We define

$P^{\mathfrak{N}_u}=P^{I_u}$	P is a predicate symbol in $\mathcal{P}$
$P^{\mathfrak{N}_u}_{[\psi]_-}=\big\{d\in\mathcal{D}\mid\mathfrak{M},z\mapsto d,u\models[\psi]_z\big\}$	ψ is a $(\bigcirc/\bullet/\ominus/\mu/\nu)$-$\mathcal{FOTL}_{\mathbf{1}\mu\nu}[\sigma]$-formula with one free individual variable
$p^{\mathfrak{N}_u}_\chi=\begin{cases}1, & \text{if } \mathfrak{M},u\models\chi\\ 0, & \text{if } \mathfrak{M},u\not\models\chi\end{cases}$	χ is a $(\bigcirc/\bullet/\ominus/\mu/\nu)$-$\mathcal{FOTL}_{\mathbf{1}\mu\nu}[\sigma]$-dom-sentence
$c^{\mathfrak{N}_u}=c^{I_u}$	c is a constant in $\mathcal{C}$

Claim 53. For any $\mathcal{FOTL}_{\mathbf{1}\mu\nu}[\sigma]$-formula ϕ, any variable assignment $h:\mathcal{V}\to\mathcal{D}$, and any moment $u\in\mathbb{N}$, if ϕ is an fp-sentence, then

$$\mathfrak{M},h,u\models\phi\iff\mathfrak{N}_u,h\models\overline{\phi}\ .$$

Proof. First, let us observe that for any term t (remember that we have no function symbols and hence the terms are just individual variables or constants) $[\![t]\!]^{I_u}_h=[\![t]\!]^{\mathfrak{N}_u}_h$. If t is a variable, then $[\![t]\!]^{I_u}_h=h(t)=[\![t]\!]^{\mathfrak{N}_u}_h$. If t is a constant, then $[\![t]\!]^{I_u}_h=t^{I_u}=t^{\mathfrak{N}_u}=[\![t]\!]^{\mathfrak{N}_u}_h$. We prove the claim by induction on ϕ.

- $\phi=p$. We have that $\mathfrak{M},h,u\models p\iff p^{I_u}=1\iff p^{\mathfrak{N}_u}=1\iff\mathfrak{N}_u,h\models p=\overline{p}$.
- $\phi=P(t_1,\dots,t_n)$. We have that $\mathfrak{M},h,u\models P(t_1,\dots,t_n)\iff([\![t_1]\!]^{I_u}_h,\dots,[\![t_n]\!]^{I_u}_h)\in P^{I_u}\iff([\![t_1]\!]^{\mathfrak{N}_u}_h,\dots,[\![t_n]\!]^{\mathfrak{N}_u}_h)\in P^{\mathfrak{N}_u}\iff\mathfrak{N}_u,h\models P(t_1,\dots,t_n)=\overline{\phi}$.
- $\phi=X$. Holds vacuously.
- The cases $\phi=\neg\phi_1,(\phi_1\wedge\phi_2),(\phi_1\vee\phi_2),\forall x\phi_1,\exists x\phi_1$ are easy.
- $\phi=\mu X\phi_1$. Since ϕ is monodic, it has at most one free individual variable.
  - Assume that ϕ has one free variable, namely y. We then have that $\mathfrak{M},h,u\models\phi=\mu X\phi_1\iff\mathfrak{M},z\mapsto h(y),u\models[\mu X\phi_1]_z\iff h(y)\in P^{\mathfrak{N}_u}_{[\mu X\phi_1]_-}\iff\mathfrak{N}_u,h\models P_{[\mu X\phi_1]_-}(y)=\overline{\mu X\phi_1}=\overline{\phi}$.

– Assume that ϕ is a sentence. We then have that $\mathfrak{M}, h, u \models \phi = \mu X\phi_1 \iff p^{\mathfrak{N}_u}_{\mu X\phi_1} = 1 \iff \mathfrak{N}_u, h \models p_{\mu X\phi_1} = \overline{\mu X\phi_1} = \overline{\phi}$.

- The cases $\phi = \bigcirc\phi_1, \newmoon\phi_1, \ominus\phi_1, \nu X\phi_1$ are similar to the previous one. □

Using the above claim, we easily deduce that

$$t^u_a = \{\psi \in \mathsf{cl}_1[\phi] \mid \mathfrak{M}, z \mapsto a, u \models [\psi]_z\} = \{\psi \in \mathsf{cl}_1[\phi] \mid \mathfrak{N}_u, z \mapsto a \models [\overline{\psi}]_z\} = t^{\mathfrak{N}_u}(a) .$$

It immediately follows that

$$T_u = \{t^u_a = t^{\mathfrak{N}_u}(a) \mid a \in \mathcal{D}\} \qquad\qquad T^{con}_u(c) = t^u_{c^{I_u}} = t^{\mathfrak{N}_u}(c^{I_u}) = t^{\mathfrak{N}_u}(c^{\mathfrak{N}_u})$$

and hence $f(u) = (T_u, T^{con}_u)$ is realized by $\mathfrak{N}_u$. So, f is a state function for ϕ. For every $a \in \mathcal{D}$, we define a function r_a from $\mathbb{N}$ to $\bigcup_{u\in\mathbb{N}} T_u$ as $r_a(u) = t^u_a$, for all $u \in \mathbb{N}$. It is easy to see that every r_a is a run in f. Let $\mathscr{R}$ be the set of all these runs, i.e. $\mathscr{R} = \{r_a \mid a \in \mathcal{D}\}$. Then, $(f, \mathscr{R})$ is obviously a quasimodel that satisfies ϕ.

Let $(\psi_1 \vee \psi_2, r_a, u)$ be the occurrence of a disjunction in $\mathfrak{m}$. We define a choice function τ that picks the disjunct ψ_i that has the $<_\phi$-least μ_ϕ-signature at the moment u under $\mathfrak{M}, h$, where $h(x) = a$ for all $x \in \mathcal{V}$.

It remains to show that the regeneration relations are converse well-founded. Consider a run $r_a \in \mathscr{R}$, a least fixpoint fp-sentence $\mu X\psi \in \mathsf{cl}_1[\phi]$, and its regeneration relation R. We show that R is converse well-founded. Assume for contradiction that R has an infinite R-ascending chain $u_0Ru_1Ru_2R\ldots$ We argue that regeneration strictly decreases μ_ϕ-signature of the occurrences under $\mathfrak{M}, h$. The first derivation step unavoidably involves unfolding the least fixpoint sentence

$$\mu X\psi_X, r_a, u_i \vdash_{\mathfrak{m},\tau} [\dddot{\psi}_X]\{\mu X\psi_X/X\}, r_a, u_i$$

and strictly decreases μ_ϕ-signature under $\mathfrak{M}, h$ by reducing the ordinal for X and by not increasing the more significant ordinals, i.e. those that correspond to fixpoint variables in $\mu\text{-}\mathsf{vars}[\psi_X]$ (see Proposition 32). Derivation steps from conjunctions, and sentences starting with $\bigcirc, \newmoon, \ominus$ cannot increase μ_ϕ-signature. In the case of disjunctions, μ_ϕ-signature does not increase, since the choice function τ always selects the disjunct with $<_\phi$-least μ_ϕ-signature under $\mathfrak{M}, h$. A derivation step from a least fixpoint sentence $\mu Y\psi_Y$ (remember that $\mu X\psi_X$ is a sub-fp-sentence of $\mu Y\psi_Y$) does not increase any ordinal up to the ordinal for X. The same holds for a derivation step from a greatest fixpoint sentence. So, the decrease of the ordinal for X cannot be canceled and none of the more significant ordinals increases. Thus, we get an infinite $<_\phi$-descending chain

$$[\mu_\phi\text{-signature of } (\mu X\psi, r_a, u_0) \text{ under } \mathfrak{M}, h] >_\phi [\mu_\phi\text{-signature of } (\mu X\psi, r_a, u_1) \text{ under } \mathfrak{M}, h] >_\phi \ldots$$

This is a contradiction, since μ_ϕ-signatures are well-ordered by $<_\phi$. □

We present now a simple example that shows why in the definition of the regeneration of a fixpoint fp-sentence $\mu X\psi_X$ we impose the restriction that $\mu X\psi_X$ is a sub-fp-sentence of all the fp-sentences in the sequence. Consider the fp-sentence $\phi_1 = \nu X\mu Y[(Px \vee \newmoon Y) \wedge \bigcirc X]$, the variable assignment h that maps x to a, and the structure $\mathfrak{M} = (\langle\mathbb{N}, <\rangle, \mathcal{D}, I)$ that interprets P as $\emptyset$ at all moments except for 0, where $P^{I_0} = \{a\}$. It is easy to verify that ϕ_1 is true at all moments under $\mathfrak{M}, h$. By unfolding ϕ_1 and renaming fixpoint variables appropriately, we get $\phi_2 = \mu Y'[(Px \vee \newmoon Y') \wedge \bigcirc\phi_1]$. It is clear that $V_\mu(\phi) = \{Y, Y'\}$. Consider the adorned quasimodel $\mathfrak{m}$ defined as in Theorem 52. The choice function is defined as one would expect, i.e. $\tau(Px \vee \newmoon\phi_2, r, 0) = Px$ and $\tau(Px \vee \newmoon\phi_2, r, u) = \newmoon\phi_2$ for $u > 0$. See Figure 2 and observe that the "regeneration" of ϕ_2 from 1 to 2 (ϕ_2 is not a sub-fp-sentence of all the fp-sentences in the sequence) along the right subtree does not decrease μ_{ϕ_1}-signature.

6.3. **"Blowing-up" first-order structures.** Before continuing with the main theorem of this section, we will prove a useful model-theoretic lemma that allows us to "blow-up" first-order structures, when the first-order language does not include equality.

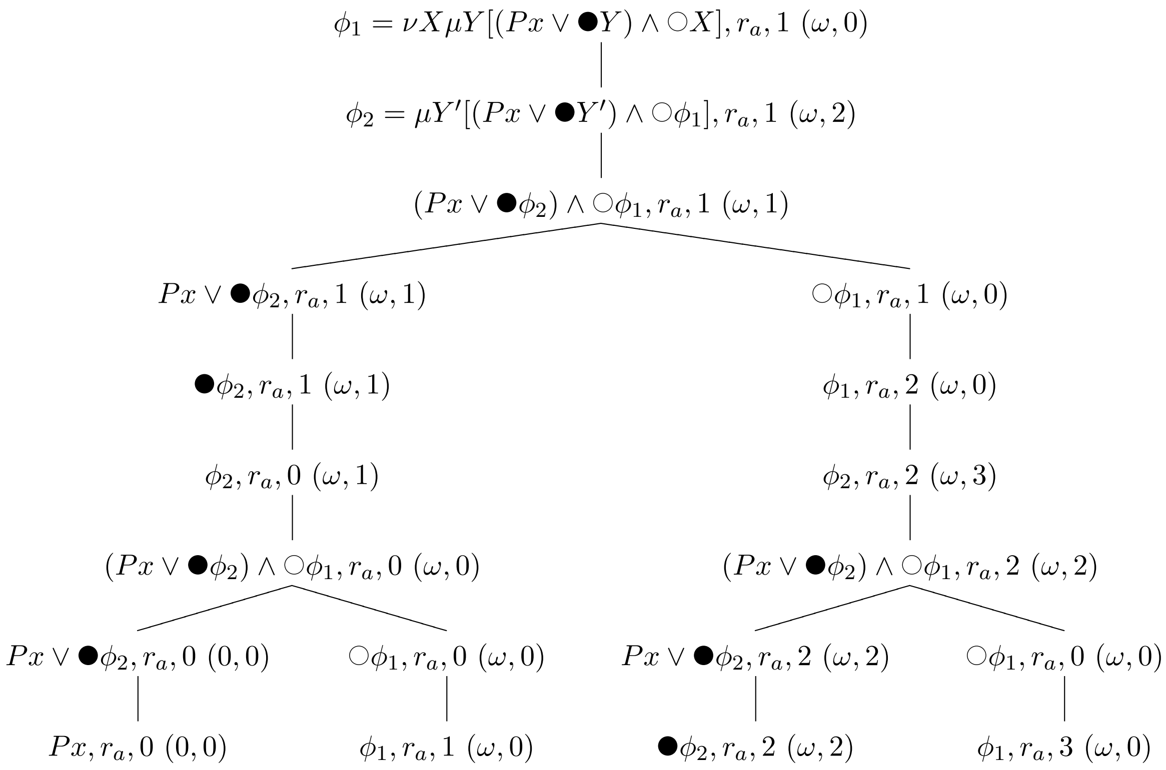

FIGURE 2. Derivations sequences and μ_{ϕ_1}-signatures. (α, β) abbreviates $\{(Y, \alpha), (Y', \beta)\}$.

Lemma 54. Let $\sigma = (\mathcal{P}, \mathcal{P}_0, \mathcal{F}, \mathcal{C}, \mathsf{ar})$ be a first-order signature. Let $\mathfrak{D} = (\mathcal{D}, \cdot^{\mathfrak{D}})$ and $\mathfrak{E} = (\mathcal{E}, \cdot^{\mathfrak{E}})$ be first-order structures over σ. Suppose that there is a relation $R \subseteq \mathcal{D} \times \mathcal{E}$ that is left-total, injective, and right total (and hence R^{-1} is a surjective function from $\mathcal{E}$ to $\mathcal{D}$). Also assume that the following hold.

(i) For any predicate symbol $P \in \mathcal{P}$ with $n = \mathsf{ar}[P]$ and any $(b_1, \ldots, b_n) \in \mathcal{E}^n$,
$$(b_1, \ldots, b_n) \in P^{\mathfrak{E}} \iff (R^{-1}(b_1), \ldots, R^{-1}(b_n)) \in P^{\mathfrak{D}} \ .$$
(ii) For any propositional variable $p \in \mathcal{P}_0$, $p^{\mathfrak{E}} = p^{\mathfrak{D}}$.
(iii) For any function symbol $f \in \mathcal{F}$ with $m = \mathsf{ar}[f]$ and any $(b_1, \ldots, b_n) \in \mathcal{E}^m$,
$$f^{\mathfrak{D}}(R^{-1}(b_1), \ldots, R^{-1}(b_m)) R f^{\mathfrak{E}}(b_1, \ldots, b_m) \ .$$
(iv) For any individual constant $c \in \mathcal{C}$, $c^{\mathfrak{D}} R c^{\mathfrak{E}}$.

Consider the variable assignments $h : \mathcal{V} \to \mathcal{D}$, $g : \mathcal{V} \to \mathcal{E}$. We define
$$hRg \overset{\text{def}}{\iff} \mathsf{Domain}(h) = \mathsf{Domain}(g) = D \text{ and } h(x)Rg(x) \text{ for all } x \in D.$$
The relation R induces an equivalence relation $\sim_R$ on $\mathcal{E}$, defined as
$$b_1 \sim_R b_2 \overset{\text{def}}{\iff} \text{there is } a \in \mathcal{D} \text{ s.t. } aRb_1 \text{ and } aRb_2 \iff R^{-1}(b_1) = R^{-1}(b_2) \ .$$
Consider the variable assignments $g_1 : \mathcal{V} \to \mathcal{E}$ and $g_2 : \mathcal{V} \to \mathcal{E}$. We define
$$g_1 \sim_R g_2 \overset{\text{def}}{\iff} \text{for all } x \in \mathcal{V},\ g_1(x) \sim_R g_2(x) \ .$$
The following hold.

(1) For any term $t \in \mathcal{T}[\sigma]$, and any $h : \mathcal{V} \to \mathcal{D}$, $g : \mathcal{V} \to \mathcal{E}$ with hRg, $[\![t]\!]^{\mathfrak{D}}_h R [\![t]\!]^{\mathfrak{E}}_g$.
(2) For any $\phi \in \mathcal{FOL}[\sigma]$, and any $h : \mathcal{V} \to \mathcal{D}$, $g : \mathcal{V} \to \mathcal{E}$ with hRg, $\mathfrak{D}, h \models \phi \iff \mathfrak{E}, g \models \phi$.
(3) For any $\phi \in \mathcal{FOL}[\sigma]$, and any $g_1 : \mathcal{V} \to \mathcal{E}$, $g_2 : \mathcal{V} \to \mathcal{E}$ with $g_1 \sim_R g_2$, $\mathfrak{E}, g_1 \models \phi \iff \mathfrak{E}, g_2 \models \phi$.

Proof. We show (1) by induction on t.

- $t = x$. Fix h, g with hRg. $[\![x]\!]^{\mathfrak{D}}_h = h(x)Rg(x) = [\![x]\!]^{\mathfrak{E}}_g$.
- $t = c$. Fix h, g with hRg. $[\![c]\!]^{\mathfrak{D}}_h = c^{\mathfrak{D}} R c^{\mathfrak{E}} = [\![c]\!]^{\mathfrak{E}}_g$.
- $t = f(t_1, \dots, t_m)$. From the inductive hypothesis we have that $[\![t_i]\!]^{\mathfrak{D}}_h R [\![t_i]\!]^{\mathfrak{E}}_g$ for all $i = 1, \dots, m$. Equivalently, $[\![t_i]\!]^{\mathfrak{D}}_h = R^{-1}([\![t_i]\!]^{\mathfrak{E}}_g)$ for all $i = 1, \dots, m$. It follows that

$$[\![f(t_1, \dots, t_m)]\!]^{\mathfrak{D}}_h = f^{\mathfrak{D}}([\![t_1]\!]^{\mathfrak{D}}_h, \dots, [\![t_m]\!]^{\mathfrak{D}}_h) = f^{\mathfrak{D}}(R^{-1}([\![t_1]\!]^{\mathfrak{E}}_g), \dots, R^{-1}([\![t_m]\!]^{\mathfrak{E}}_g))R \\ f^{\mathfrak{E}}([\![t_1]\!]^{\mathfrak{E}}_g, \dots, [\![t_m]\!]^{\mathfrak{E}}_g) = [\![f(t_1, \dots, t_m)]\!]^{\mathfrak{E}}_g \ .$$

We show (2) by induction on ϕ.

- $\phi = p$. Fix h, g with hRg. $\mathfrak{D}, h \models p \iff p^{\mathfrak{D}} = 1 \iff p^{\mathfrak{E}} = 1 \iff \mathfrak{E}, g \models p$.
- $\phi = P(t_1, \dots, t_n)$. Fix h, g with hRg. From (1) we have that $[\![t_i]\!]^{\mathfrak{D}}_h R [\![t_i]\!]^{\mathfrak{E}}_g$ for all $i = 1, \dots, n$. It follows that

$$\mathfrak{D}, h \models P(t_1, \dots, t_n) \iff ([\![t_1]\!]^{\mathfrak{D}}_h, \dots, [\![t_n]\!]^{\mathfrak{D}}_h) \in P^{\mathfrak{D}} \iff (R^{-1}([\![t_1]\!]^{\mathfrak{E}}_g), \dots, R^{-1}([\![t_n]\!]^{\mathfrak{E}}_g)) \in P^{\mathfrak{D}} \\ \iff ([\![t_1]\!]^{\mathfrak{E}}_g, \dots, [\![t_n]\!]^{\mathfrak{E}}_g) \in P^{\mathfrak{E}} \iff \mathfrak{E}, g \models P(t_1, \dots, t_n) \ .$$

- The cases $\phi = \neg\phi_1$ and $\phi = (\phi_1 \wedge \phi_2)$ are easy.
- $\phi = \exists x \phi_1$. Fix h, g with hRg.
  - Suppose that $\mathfrak{D}, h \models \exists x \phi_1$. There is $a \in \mathcal{D}$ such that $\mathfrak{D}, h[x \mapsto a] \models \phi_1$. Since R is left-total, there is $b \in \mathcal{E}$ such that aRb. Observe that $h[x \mapsto a]Rg[x \mapsto b]$. By the inductive hypothesis, $\mathfrak{E}, g[x \mapsto b] \models \phi_1$, which implies that $\mathfrak{E}, g \models \exists x \phi_1$.
  - Suppose that $\mathfrak{E}, g \models \exists x \phi_1$, which means that there is $b \in \mathcal{E}$ such that $\mathfrak{E}, g[x \mapsto b] \models \phi_1$. Let $a = R^{-1}(b)$ and observe that $h[x \mapsto a]Rg[x \mapsto b]$. From the inductive hypothesis we get that $\mathfrak{D}, h[x \mapsto a] \models \phi_1$ and hence $\mathfrak{D}, h \models \exists x \phi_1$.

(3) is an easy consequence of (2). Fix $g_1 : \mathcal{V} \to \mathcal{E}$, $g_2 : \mathcal{V} \to \mathcal{E}$, and define $h : \mathcal{V} \to \mathcal{D}$ as $h(x) = R^{-1}(g_1(x)) = R^{-1}(g_2(x))$ for all $x \in \mathcal{V}$. Observe that hRg_1 and hRg_2. Therefore,

$$\mathfrak{E}, g_1 \models \phi \iff \mathfrak{D}, h \models \phi \iff \mathfrak{E}, g_2 \models \phi \ . \qquad \square$$

Remark 55. Let us see why the proof of Lemma 54 cannot be extended when we include equality in the first-order language. Consider two first-order structures $\mathfrak{D} = (\mathcal{D}, \cdot^{\mathfrak{D}})$, $\mathfrak{E} = (\mathcal{E}, \cdot^{\mathfrak{E}})$, and a relation $R \subseteq \mathcal{D} \times \mathcal{E}$ that satisfy the conditions of Lemma 54. It is possible that there is an element a of $\mathcal{D}$ that is related through R to two distinct elements b_1, b_2 of $\mathcal{E}$. That is, $b_1 \neq b_2$, aRb_1, and aRb_2. Observe that the partial variable assignments $\{(x, a), (y, a)\}$ and $\{(x, b_1), (y, b_2)\}$ can be easily extended to total variable assignments h and g respectively such that hRg. Then, we would have that $\mathfrak{D}, h \models x = y$ and $\mathfrak{E}, g \not\models x = y$.

6.4. **Satisfaction in well-founded quasimodel implies satisfaction in model.** Suppose that ϕ is satisfied in some well-founded adorned quasimodel $\mathfrak{m} = (f, \mathscr{R}, \tau)$. We will show that we can construct from $\mathfrak{m}$ a temporal structure that satisfies ϕ. First, we argue by the Löwenheim-Skolem theorem that each quasistate $f(u)$ is realized by some first-order structure $\mathfrak{D}_u = (\mathcal{D}_u, \cdot^{\mathfrak{D}_u})$ of countable domain $\mathcal{D}_u$. Clearly, we cannot construct a temporal model from this sequence of structures because, for one thing, they have different domains. The idea is to blow these structures up by virtue of Lemma 54 so that they share the same domain and so that constants are rigidly interpreted. Take a cardinal $\kappa \geq \aleph_0$. κ is bigger than any $\mathcal{D}_u$. We define $\mathcal{E} = \mathscr{R} \times \kappa$. For each moment u, we define appropriately a left-total, injective, and right-total relation $R_u \subseteq \mathcal{D}_u \times \mathcal{E}$. Lemma 54 allows us to define a family of first-order structures $\{\mathfrak{E}_u\}_{u \in \mathbb{N}}$ with the same domain $\mathcal{E}$ that realize the corresponding quasistates. We have also arranged that constants are rigidly interpreted. We string the structures $\{\mathfrak{E}_u\}_{u \in \mathbb{N}}$ together into a first-order temporal structure $\mathfrak{M}$.

The crucial claim is that if a formula $\overline{\psi}$, with ψ in the closure of ϕ, is satisfied at some moment u under the structure $\mathfrak{E}_u$ and some individual variable assignment h, then ψ is satisfied under $\mathfrak{M}, h$ at moment u. We consider such triples (ψ, h, u) and gather them under the set $\textsc{Occ}(\mathfrak{E})$. It is rather straightforward to extend the derivation relation to $\textsc{Occ}(\mathfrak{E})$ and argue about the converse well-foundedness of the regeneration relations. Due to the monodic restrictions, the regeneration of least fixpoint formulas cannot involve formulas with more than one free variables

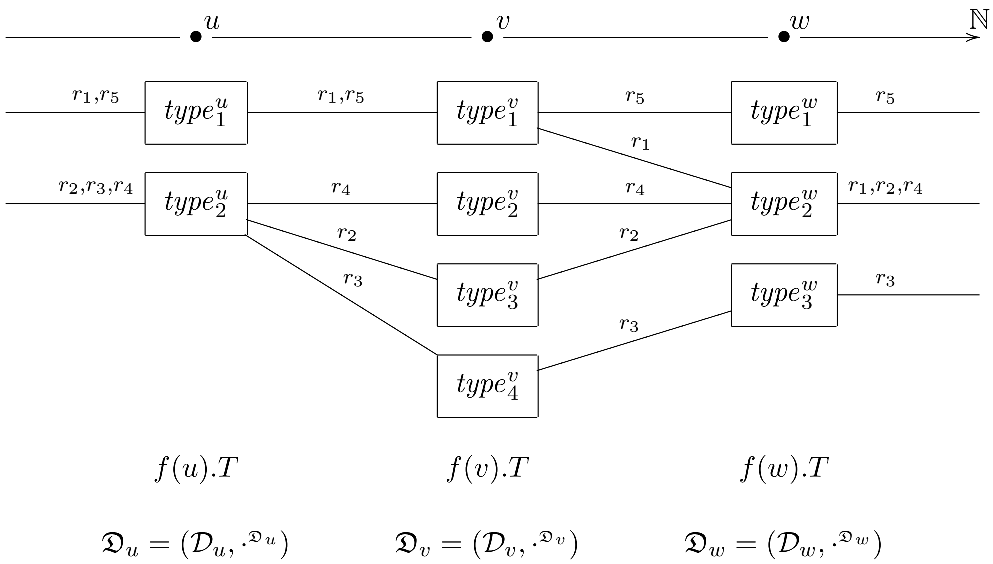

FIGURE 3. Runs, quasistates and first-order structures that realize them.

nor quantified formulas. To each occurrence of a least fixpoint formula $\mu X \psi_X$ in $\mathrm{OCC}(\mathfrak{E})$, we assign the "well-ordering ordinal" (in the words of Streett and Emerson) of the regeneration relation for $\mu X \psi_X$. Then, we can attach an appropriate μ-annotation, which is reminiscent of signatures but not the same thing, to each occurrence (ψ, h, u) in $\mathrm{OCC}(\mathfrak{E})$ and proceed to show the claim by a delicate induction on these annotations and formula structure.

Theorem 56. Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence. If ϕ is satisfied in some well-founded adorned quasimodel for ϕ, then it is satisfiable.

Proof. Fix a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence ϕ and suppose that ϕ is satisfied in some well-founded adorned quasimodel $(f, \mathscr{R}, \tau)$ for ϕ. There is a moment $w \in W$ such that ϕ is in all types in $f(w).T$.

Let u be an arbitrary moment. By definition, $f(u)$ is a realizable state candidate. It follows that the first-order sentence $\alpha_{f(u)}$, defined as in Lemma 46, is satisfiable. By the Löwenheim-Skolem theorem [39], there exists a first-order structure $\mathfrak{D}_u = (\mathcal{D}_u, \cdot^{\mathfrak{D}_u})$ over the signature σ_{surr} with countable domain $\mathcal{D}_u$ that satisfies $\alpha_{f(u)}$ and hence also realizes $f(u)$ (see proof of Lemma 46). That is,

$$f(u).T = \{t^{\mathfrak{D}_u}(a) \mid a \in \mathcal{D}_u\} \qquad f(u).T^{con}(c) = t^{\mathfrak{D}_u}(c^{\mathfrak{D}_u}) \text{ , for all } c \in \mathsf{con}[\phi] \text{ .}$$

Remember from Remark 43 that the set of types $f(u).T$ defines an equivalence relation and thus partitions the domain $\mathcal{D}_u$. Clearly, each equivalence class

$$\mathfrak{D}_{u[t]} = \{a \in \mathcal{D}_u \mid t = t^{\mathfrak{D}_u}(a)\}$$

is of cardinality at most $\aleph_0$. See Figure 3. Take a cardinal $\kappa \geq \aleph_0$. We will construct a model with domain $\mathcal{E}$ defined as

$$\mathcal{E} = \mathscr{R} \times \kappa = \{(r, \xi) \mid r \in \mathscr{R} \text{ and } \xi \in \kappa\} \text{ .}$$

It is clear that $\mathcal{E}$ is of cardinality κ. The challenge now is how to deal with the fact that we have first-order structures of different domains. We will construct a family of appropriate left-total, injective, and right-total relations $\{R_u\}_{u \in \mathbb{N}}$ from the domains $\{\mathcal{D}_u\}_{u \in \mathbb{N}}$ to $\mathcal{E}$. By virtue of Lemma 54, we will then get a family of first-order structures $\{\mathfrak{E}_u\}_{u \in W}$, with $\mathfrak{E}_u = (\mathcal{E}, \cdot^{\mathfrak{E}_u})$, that realize the quasistates and that all share the same domain. Bear also in mind that the constants have to be rigidly interpreted, so these relations have to be carefully defined.

Fix a type t in $f(u).T$ and observe that $\mathfrak{D}_{u[t]}$ may contain the interpretation of some constants. Let C be the set of these constants and D the set of their interpretations, i.e.

$$C = \{c \in \mathcal{C} \mid c^{\mathfrak{D}_u} \in \mathfrak{D}_{u[t]}\} \qquad D = \{d \in \mathfrak{D}_{u[t]} \mid \text{there is } c \in \mathcal{C} \text{ s.t. } d = c^{\mathfrak{D}_u}\} \text{ .}$$

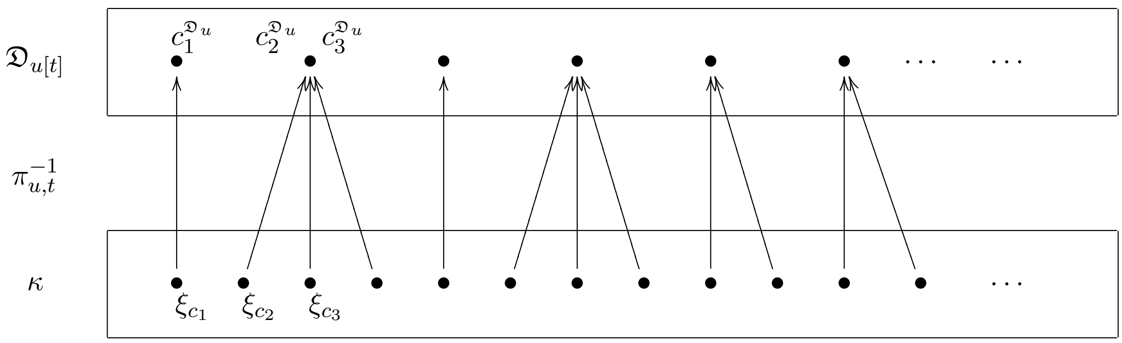

FIGURE 4. A left-total, injective, and right-total relation $\pi_{u,t}$ from $\mathfrak{D}_{u[t]}$ to κ.

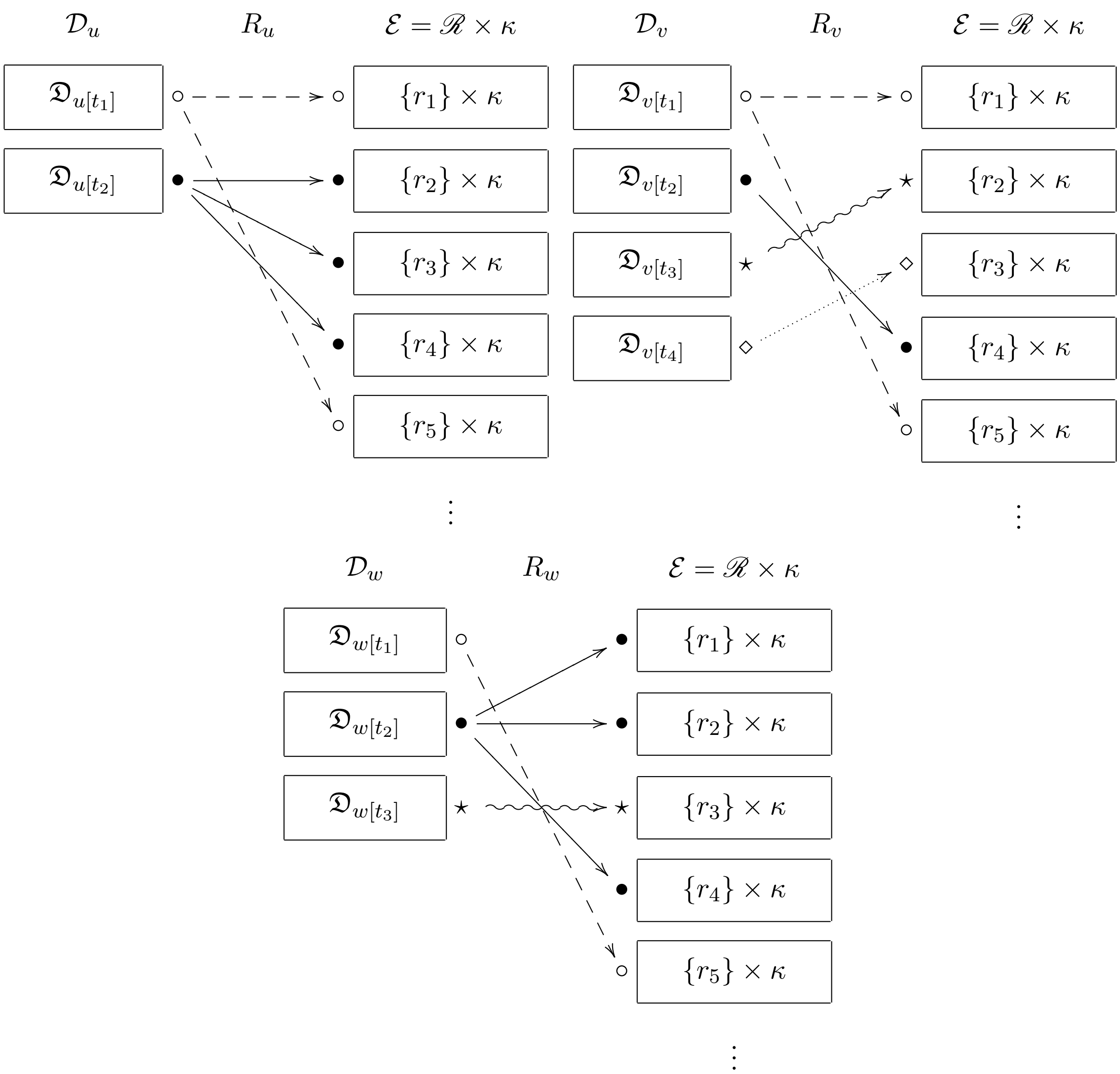

FIGURE 5. Constructing piecewise the relation R_u from $\mathcal{D}_u$ to $\mathcal{E}$.

Without loss of generality we can assume that the constants that the signature contains are only those that appear in ϕ, i.e. $\mathcal{C} = \mathsf{con}[\phi]$. We fix an injection that maps each constant $c \in \mathcal{C}$ to a distinct $\xi_c \in \kappa$. We define a left-total, injective, and right-total relation $\pi_{u,t}$ from $\mathfrak{D}_{u[t]}$ to κ or, equivalently, a surjective function $\pi_{u,t}^{-1}$ from κ to $\mathfrak{D}_{u[t]}$ as follows.

– For every constant c in C, $\pi_{u,t}^{-1}(\xi_c) = c^{\mathfrak{D}_u}$.
– Observe that $\kappa' = \kappa \setminus \{\xi_c \mid c \in C\}$ is of cardinality κ, which means that it has "much more" elements than $\mathfrak{D}_{u[t]}$. So, we can map each element in κ' to some element in $\mathfrak{D}_{u[t]}$ and arrange that every element in $\mathfrak{D}_{u[t]}$ is mapped to.

See Figure 4 for an illustration of the idea.

Each type in $f(u).T$ has a run that goes through it and all runs go through some type in $f(u).T$ at time u. Hence, to each type in $f(u).T$ we associate the non-empty set of runs that go through it at u. So, we define a family of functions $\{\mathsf{runs}_u\}_{u\in\mathbb{N}}$ as follows.

$$\mathsf{runs}_u : f(u).T \to \wp(\mathscr{R}) \qquad\qquad \mathsf{runs}_u(t) = \{r \in \mathscr{R} \mid r(u) = t\}$$

For the example of Figure 3 we would have $\mathsf{runs}_u(t_1) = \{r_1, r_5, \ldots\}$ (we put the dots because we obviously have not drawn all the runs in the figure) and $\mathsf{runs}_u(t_2) = \{r_2, r_3, r_4, \ldots\}$. Let us state three obvious facts about these mappings.

$$t \neq t' \implies \mathsf{runs}_u(t) \cap \mathsf{runs}_u(t') = \emptyset \qquad \bigcup_{t\in f(u).T} \mathsf{runs}_u(t) = \mathscr{R} \quad \mathcal{E} = \mathscr{R} \times \kappa = \bigcup_{t\in f(u).T} \big[\mathsf{runs}_u(t) \times \kappa\big]$$

We define the left-total, injective, and right-total relations $R_{u,t} \subseteq \mathfrak{D}_{u[t]} \times \big[\mathsf{runs}_u(t) \times \kappa\big]$, and the relation $R_u \subseteq \mathcal{D}_u \times \mathcal{E}$ as

$$dR_{u,t}(r,\xi) \overset{\text{def}}{\iff} r \in \mathsf{runs}_u(t) \text{ and } (d,\xi) \in \pi_{u,t} \qquad\qquad R_u = \bigcup_{t\in f(u).T} R_{u,t}$$

It is easy to see that R_u is a left-total, injective, and right-total relation from $\mathcal{D}_u$ to $\mathcal{E}$.

Now, we define a family of first-order structures $\{\mathfrak{E}_u\}_{u\in\mathbb{N}}$ over the signature σ_{surr} that have the same domain $\mathcal{E}$. The interpretation function $\cdot^{\mathfrak{E}_u}$ is defined so that it satisfies the following.

(i) $(b_1,\ldots,b_n) \in P^{\mathfrak{E}_u} \iff (R_u^{-1}(b_1),\ldots,R_u^{-1}(b_n)) \in P^{\mathfrak{D}_u}$,
(ii) $p^{\mathfrak{E}_u} = p^{\mathfrak{D}_u}$,
(iii) $c^{\mathfrak{E}_u} = (r_c,\xi_c)$, which implies that $c^{\mathfrak{D}_u} R_u c^{\mathfrak{E}_u}$, for all $c \in \mathsf{con}[\phi]$. Notice that constants are interpreted rigidly.

Observe that the conditions of Lemma 54 for the first-order structures $\mathfrak{D}_u$ and $\mathfrak{E}_u$ are met. We proceed to show that $t^{\mathfrak{E}_u}(b) = t^{\mathfrak{D}_u}(R_u^{-1}(b))$. Let $\psi \in \mathsf{cl}_1[\phi]$. We have that

$$\psi \in t^{\mathfrak{E}_u}(b) \iff \mathfrak{E}_u, z \mapsto b \models [\overline{\psi}]_z \overset{\text{Lemma 54}}{\iff} \mathfrak{E}_u, z \mapsto R_u^{-1}(b) \models [\overline{\psi}]_z \iff \psi \in t^{\mathfrak{D}_u}(R_u^{-1}(b)) \ .$$

It is easy to show that $\mathfrak{E}_u$ realizes $f(u)$.

- Let t be a type in $f(u).T$. Since $f(u)$ is realized by $\mathfrak{D}_u$, there is $a \in \mathcal{D}_u$ such that $t = t^{\mathfrak{D}_u}(a)$. R_u is left-total, so there is $b \in \mathcal{E}$ such that aR_ub. It follows that $t^{\mathfrak{E}_u}(b) = t^{\mathfrak{D}_u}(R_u^{-1}(b)) = t^{\mathfrak{D}_u}(a) = t$.
- Let $b \in \mathcal{E}$. Then, $t^{\mathfrak{E}_u}(b) = t^{\mathfrak{D}_u}(R_u^{-1}(b))$. Since $R_u^{-1}(b) \in \mathcal{D}_u$ and $\mathfrak{D}_u$ realizes $f(u)$, we get that $t^{\mathfrak{D}_u}(R_u^{-1}(b))$ is in $f(u).T$.
- Let $c \in \mathsf{con}[\phi]$. $\mathfrak{D}_u$ realizes $f(u)$ and hence $f(u).T^{con}(c) = t^{\mathfrak{D}_u}(c^{\mathfrak{D}_u})$. Observe that $c^{\mathfrak{D}_u} R_u c^{\mathfrak{E}_u}$, which implies that $t^{\mathfrak{E}_u}(c^{\mathfrak{E}_u}) = t^{\mathfrak{D}_u}(R_u^{-1}(c^{\mathfrak{E}_u})) = t^{\mathfrak{D}_u}(c^{\mathfrak{D}_u}) = f(u).T^{con}(c)$.

Observe the following properties for $\psi \in \mathsf{cl}_1[\phi]$.

(1) $\psi \in r(u) \implies$ for all $\xi \in \kappa$, $\mathfrak{E}_u, z \mapsto (r,\xi) \models [\overline{\psi}]_z$.
Suppose that $\psi \in r(u)$. Let $t = r(u)$, $\xi \in \kappa$, and $a = \pi_{u,t}^{-1}(\xi) \in \mathfrak{D}_{u[t]}$. Then, $t = r(u) = t^{\mathfrak{D}_u}(a)$, $r \in \mathsf{runs}_u(t)$, and $aR_{u,t}(r,\xi)$. We have that $\mathfrak{D}_u, z \mapsto a \models [\overline{\psi}]_z$ and hence $\mathfrak{E}_u, z \mapsto (r,\xi) \models [\overline{\psi}]_z$.
(2) $\mathfrak{E}_u, z \mapsto (r,\xi) \models [\overline{\psi}]_z \implies \psi \in r(u)$.
Suppose that $\mathfrak{E}_u, z \mapsto (r,\xi) \models [\overline{\psi}]_z$. Let $a = R_u^{-1}(r,\xi)$ and $t = t^{\mathfrak{D}_u}(a)$, which means that $aR_u(r,\xi)$ and $r \in \mathsf{runs}_u(t)$. It follows that $\mathfrak{D}_u, z \mapsto a \models [\overline{\psi}]_z$ and hence $\psi \in t^{\mathfrak{D}_u}(a) = r(u)$.

Consider now the first-order temporal structure $\mathfrak{M} = \big(\langle\mathbb{N},<\rangle, \mathcal{E}, I\big)$, where $I_w = \mathfrak{E}_w = (\mathcal{E}, \cdot^{\mathfrak{E}_w})$. I_w interprets the symbols of σ_{surr}, which includes the symbols of σ.

First, let us introduce a new set of occurrences that also includes fp-sentences from $\mathsf{cl}[\phi]$ that have more than one free individual variable. We define

$$\textsc{Occ}(\mathfrak{E}) = \big\{(\psi, h, u) \mid u \in \mathbb{N},\ \psi \in \mathsf{cl}[\phi],\ h : \mathsf{fvars}[\psi] \to \mathscr{R} \times \kappa,\ \mathfrak{E}_u, h \models \overline{\psi}\big\}\ .$$

It is easy to see that for every occurrence (ψ, r, u) in $Occ(\mathfrak{m})$, where ψ has one free individual variable, say x, there are corresponding occurrences $(\psi, \{x \mapsto (r, \xi)\}, u)$ in $\textsc{Occ}(\mathfrak{E})$ for every $\xi \in \kappa$. For all occurrences $(\psi, r, u)_{r\in\mathscr{R}}$, where ψ is a dom-sentence, we have only one occurrence $(\psi, \emptyset, u)$ in $\textsc{Occ}(\mathfrak{E})$.

Now, we have to extend the choice function τ. The extended choice function picks a disjunct for each disjunction occurrence. Moreover, it picks for an existentially quantified formula a pair (r, ξ) for the variable that the quantifer binds. Consider an occurrence $(\psi_1 \vee \psi_2, h, u)$ in $\textsc{Occ}(\mathfrak{E})$.

- If $(\psi_1 \vee \psi_2)$ is a dom-sentence, then take any run $r \in \mathscr{R}$ (it does not matter which one we take) and define $\tau(\psi_1 \vee \psi_2, \emptyset, u) = \tau(\psi_1 \vee \psi_2, r, u)$.
- If $(\psi_1 \vee \psi_2)$ has one free individual variable, say x, then $\tau(\psi_1 \vee \psi_2, h, u) = \tau(\psi_1 \vee \psi_2, h(x).r, u)$. We write $h(x).r$ to denote r when $h(x) = (r, \xi)$.
- If $(\psi_1 \vee \psi_2)$ has more than one free individual variable, then $\mathfrak{E}_u, h \models \overline{\psi_1 \vee \psi_2} = \overline{\psi_1} \vee \overline{\psi_2}$. So, for at least one i in $\{1, 2\}$, $\mathfrak{E}_u, h_i \models \overline{\psi_i}$, where h_i is the restriction of h to $\mathsf{fvars}[\psi_i]$. We define that τ picks ψ_i. Notice that if both $\overline{\psi_1}, \overline{\psi_2}$ are true, it does not matter which one we choose. The regeneration of least fixpoint fp-sentences does not involve formulas with more than one free variable.

Consider an occurrence $(\exists x\psi, h, u)$ in $\textsc{Occ}(\mathfrak{E})$. This means that $\mathfrak{E}_u, h \models \overline{\exists x\psi} = \exists x\overline{\psi}$. So, there is at least one $(r, \xi) \in \mathcal{E}$ such that $\mathfrak{E}_u, h \cup \{x \mapsto (r, \xi)\} \models \overline{\psi}$. Pick any of these (r, ξ) (the regeneration of least fixpoint fp-sentences does not involve existentially quantified formulas) and define $\tau(\exists x\psi, h, u) = (r, \xi)$.

We define the derivation relation $\vdash_{\mathfrak{E},\tau}$ on $\textsc{Occ}(\mathfrak{E})$ as follows.

(1) If $(\psi_1 \wedge \psi_2, h, u) \in \textsc{Occ}(\mathfrak{E})$, then $\psi_1 \wedge \psi_2, h, u \vdash_{\mathfrak{E},\tau} \psi_1, h_1, u$ and $\psi_1 \wedge \psi_2, h, u \vdash_{\mathfrak{E},\tau} \psi_2, h_2, u$, where h_1 (h_2) is the restriction of h to $\mathsf{fvars}[\psi_1]$ ($\mathsf{fvars}[\psi_2]$).
(2) If $(\psi_1 \vee \psi_2, h, u) \in \textsc{Occ}(\mathfrak{E})$, then $\psi_1 \vee \psi_2, h, u \vdash_{\mathfrak{E},\tau} \psi_i, h_i, u$, where $\psi_i = \tau(\psi_1 \vee \psi_2, h, u)$ and h_i is the restriction of h to $\mathsf{fvars}[\psi_i]$.
(3) If $(\forall x\psi, h, u) \in \textsc{Occ}(\mathfrak{E})$, then $\forall x\psi, h, u \vdash_{\mathfrak{E},\tau} \psi, h \cup \{x \mapsto (r, \xi)\}, u$, for all $r \in \mathscr{R}$ and all $\xi \in \kappa$.
(4) If $(\exists x\psi, h, u) \in \textsc{Occ}(\mathfrak{E})$, then $\exists x\psi, h, u \vdash_{\mathfrak{E},\tau} \psi, h \cup \{x \mapsto (r, \xi)\}, u$, where $(r, \xi) = \tau(\exists x\psi, h, u)$.
(5) If $(\bigcirc\psi, h, u) \in \textsc{Occ}(\mathfrak{E})$, then $\bigcirc\psi, h, u \vdash_{\mathfrak{E},\tau} \psi, h, u + 1$.
(6) If $(\bullet\psi, h, u) \in \textsc{Occ}(\mathfrak{E})$, then $u > 0$ (by definition) and $\bullet\psi, h, u \vdash_{\mathfrak{E},\tau} \psi, h, u - 1$.
(7) If $(\ominus\psi, h, u) \in \textsc{Occ}(\mathfrak{E})$ and $u > 0$, then $\ominus\psi, h, u \vdash_{\mathfrak{E},\tau} \psi, h, u - 1$.
(8) If $(\mu X\psi, h, u) \in \textsc{Occ}(\mathfrak{E})$, then $\mu X\psi, h, u \vdash_{\mathfrak{E},\tau} [\dddot{\psi}]\{\mu X\psi/X\}, h, u$.
(9) If $(\nu X\psi, h, u) \in \textsc{Occ}(\mathfrak{E})$, then $\nu X\psi, h, u \vdash_{\mathfrak{E},\tau} [\dddot{\psi}]\{\nu X\psi/X\}, h, u$.

The transitive closure of $\vdash_{\mathfrak{E},\tau}$ is denoted by $\vdash^+_{\mathfrak{E},\tau}$ and its reflexive transitive closure by $\vdash^*_{\mathfrak{E},\tau}$. We have already mentioned that the derivation sequence of a least fixpoint fp-sentence regeneration does not involve formulas with more than one free variable nor quantified formulas. This is because of the monodic restrictions. Consequently, we could just as well define the regeneration relations using the derivation relation $\vdash_{\mathfrak{E},\tau}$ instead of $\vdash_{\mathfrak{m},\tau}$.

Consider some least fixpoint fp-sentence $\mu X\psi_X$ that is in $\mathsf{cl}_1[\phi]$ and some $h : \mathsf{fvars}[\mu X\psi_X] \to \mathscr{R} \times \kappa$. By monodicity of $\mu X\psi_X$, h is either empty or a singleton. The regeneration relation $R^{\mathfrak{E},\tau}_{h,X}$ is defined as: $(u, v) \in R^{\mathfrak{E},\tau}_{h,X}$ iff there is a finite sequence $(\psi_1, h_1, u_1), (\psi_2, h_2, u_2), \ldots, (\psi_k, h_k, u_k)$ of occurrences in $\textsc{Occ}(\mathfrak{E})$ such that

- $\psi_1 = \mu X\psi_X$, $h_1 = h$, and $u_1 = u$,
- $\psi_k = \mu X\psi_X$, $h_k = h$, and $u_k = v$,
- $\psi_i, h_i, u_i \vdash_{\mathfrak{E},\tau} \psi_{i+1}, h_{i+1}, u_{i+1}$ for all $i \in \{1, \ldots, k - 1\}$,
- $\mu X\psi_X$ is a sub-fp-sentence of ψ_i for all $i \in \{1, \ldots, k\}$.

It is easy to see that all these relations are converse well-founded. Define the relation $R_X^{\mathfrak{E},\tau}$ on the set

$$\{(h,u) \mid u \in \mathbb{N}, h : \mathsf{fvars}[\mu X \psi_X] \to \mathscr{R} \times \kappa\}$$

as

$$(h_1,u_1) R_X^{\mathfrak{E},\tau} (h_2,u_2) \overset{\text{def}}{\iff} h_1 = h_2 = h \text{ and } (u,v) \in R_{h,X}^{\mathfrak{E},\tau}$$

and observe that it is converse well-founded. From now on, we will drop the superscript and write just R_X to mean $R_X^{\mathfrak{E},\tau}$. We can assign to each occurrence of $\mu X \psi_X$ in $\textsc{Occ}(\mathfrak{E})$ an ordinal as follows. Define

$$U_X = \big\{(h,u) \mid (\mu X \psi_X, h, u) \in \textsc{Occ}(\mathfrak{E})\big\}$$

$$B_\alpha = U_X \setminus \bigcup_{\beta<\alpha} A_\beta \quad A_\alpha = \big\{(h,u) \in B_\alpha \mid (h,u) \text{ is } R_X(B_\alpha)\text{-maximal}\big\} .$$

We write $R_X(B)$ to mean the restriction of R_X to $B \subseteq U_X$. It is easy to see that $\alpha \neq \beta \implies A_\alpha \cap A_\beta = \emptyset$. For contradiction, assume that there are ordinals $\alpha \neq \beta$ (without loss of generality $\alpha < \beta$) and a pair $(h,u) \in U_X$ such that $(h,u) \in A_\alpha$ and $(h,u) \in A_\beta$. So, $(h,u) \in B_\beta$, which implies that $(h,u) \notin A_\alpha$. Contradiction.

Claim 57. For any pair $(h,u) \in U_X$, there is an ordinal α such that $(h,u) \in A_\alpha$.

Proof. We proceed by well-founded induction on the converse well-founded relation R_X. Let $(h,u) \in U_X$ and assume that the property holds for every R_X-successor of (h,u). That is, $(h,u) R_X (h',v)$ implies that there is an ordinal $\beta(h',v)$ such that $(h',v) \in A_{\beta(h',v)}$. Assume to the contrary that there is no ordinal α such that $(h,u) \in A_\alpha$. Let $\alpha = \sup\{\beta(h',v) \mid (h,u) R_X (h',v)\}$ and notice that $(\alpha+1)$ is strictly greater than all ordinals in $\{\beta(h',v) \mid (h,u) R_X (h',v)\}$. So, no R_X-successor of (h,u) is in $B_{\alpha+1} = U_X \setminus \bigcup_{\beta<\alpha+1} A_\alpha$, but (h,u) is in $B_{\alpha+1}$. That is, (h,u) is $R_X(B_{\alpha+1})$-maximal and hence $(h,u) \in A_{\alpha+1}$. Contradiction. $\square$

Claim 58. Let $(h,u) \in U_X$ be such that $(h,u) \in A_\alpha$. Then, all R_X-successors of (h,u) are in $\bigcup_{\beta<\alpha} A_\beta$.

Proof. Since $(h,u) \in A_\alpha$, $(h,u) \in B_\alpha$ and (h,u) is $R_X(B_\alpha)$-maximal. This means that no R_X-successor of (h,u) is in B_α, because from $(h,u) R_X (g,v)$ and $(g,v) \in B_\alpha$ we would get that $(h,u) R_X(B_\alpha) (g,v)$, which is a contradiction. It follows that all R_X-successors of (h,u) are in $\bigcup_{\beta<\alpha} A_\beta$. $\square$

From the previous discussion and Claim 57 we deduce that the family of sets $\{A_\alpha\}_{\mathbf{Ord}}$ partitions U_X and we can define the function $f_X : U_X \to \mathbf{Ord}$ as

$$f_X(h,u) = [\text{the unique ordinal } \alpha \text{ for which } (h,u) \in A_\alpha] + 1 .$$

All the values of f_X are successor ordinals. An immediate consequence of Claim 58 is that for any $(h,u), (g,v) \in U_X$,

$$(h,u) R_X (g,v) \implies f_X(h,u) > f_X(g,v) .$$

To each occurrence (ψ,h,u) in $\textsc{Occ}(\mathfrak{E})$ we associate a μ-annotation $a_{\psi,h,u} : \mu\text{-}\mathsf{vars}[\phi] \to \mathbf{Ord}$ for ψ, defined as

$$a_{\psi,h,u}(X) = \sup\Big\{ f_X(g,v) \mid \psi,h,u \vdash^*_{\mathfrak{m},\tau} \mu X \psi_X, g, v \Big\} .$$

Clearly, these annotations are well-ordered by $<_\phi$. Immediately from the definition we deduce that

- If $(\psi = \psi_1 \wedge \psi_2, h, u) \in \textsc{Occ}(\mathfrak{E})$, then $a_{\psi_1,h_1,u} \leq_\phi a_{\psi,h,u}$ and $a_{\psi_2,h_2,u} \leq_\phi a_{\psi,h,u}$, where h_1 (h_2) is the restriction of h to $\mathsf{fvars}[\psi_1]$ ($\mathsf{fvars}[\psi_2]$).
- If $(\psi = \psi_1 \vee \psi_2, h, u) \in \textsc{Occ}(\mathfrak{E})$, then $a_{\psi,h,u} = a_{\psi',h',u}$, where $\psi' = \tau(\psi,h,u)$ and h' is the restriction of h to $\mathsf{fvars}[\phi']$.
- If $(\psi = \forall x \psi_1, h, u) \in \textsc{Occ}(\mathfrak{E})$, then for all $r \in \mathscr{R}$ and all $\xi \in \kappa$, $a_{\psi_1,h',u} \leq_\phi a_{\psi,h,u}$, where $h' = h \cup \{x \mapsto (r,\xi)\}$.
- If $(\psi = \exists x \psi_1, h, u) \in \textsc{Occ}(\mathfrak{E})$, then $a_{\psi,h,u} = a_{\psi_1,h',u}$, where $h' = h \cup \{x \mapsto \tau(\psi,h,u)\}$.

– If $(\psi = \bigcirc\psi_1, h, u) \in \textsc{Occ}(\mathfrak{E})$, then $a_{\psi,h,u} = a_{\psi_1,h,u+1}$.
– If $(\psi = \bullet\psi_1, h, u) \in \textsc{Occ}(\mathfrak{E})$, then $u > 0$ and $a_{\psi,h,u} = a_{\psi_1,h,u-1}$.
– If $(\psi = \ominus\psi_1, h, u) \in \textsc{Occ}(\mathfrak{E})$ and $u > 0$, then $a_{\psi,h,u} = a_{\psi_1,h,u-1}$.
– If $(\psi = \mu X\psi_X, h, u) \in \textsc{Occ}(\mathfrak{E})$, then $a_{\psi,h,u}(Y) = a_{\psi',h,u}(Y)$ for $Y \neq X$ and $a_{\psi,h,u}(X) > a_{\psi',h,u}(X)$, where $\psi' = [\dddot{\psi}_X]\{\mu X\psi_X/X\}$.

$$a_{\psi,h,u}(X) = \sup \underbrace{\left\{ f_X(h', v) \mid \mu X\psi_X, h, u \vdash^*_{\mathfrak{E},\tau} \mu X\psi_X, h', v \right\}}_{A}$$

$$a_{\psi',h,u}(X) = \sup \underbrace{\left\{ f_X(h', v) \mid [\dddot{\psi}_X]\{\mu X\psi_X/X\}, h, u \vdash^*_{\mathfrak{E},\tau} \mu X\psi_X, h', v \right\}}_{B}$$

It is easy to see that $B \subseteq A$ and $f_X(h, u) \in A$. We show that $f_X(h, u)$ is strictly greater than all ordinals in B. Let $\beta \in B$. There is $(g, w) \in U_X$ such that $\beta = f_X(g, w)$ and $[\dddot{\psi}_X]\{\mu X\psi_X/X\}, h, u \vdash^*_{\mathfrak{E},\tau} \mu X\psi_X, g, w$. It follows that $\mu X\psi_X, h, u \vdash^+_{\mathfrak{E},\tau} \mu X\psi_X, g, w$, which means that (g, w) is a R_X-successor of (h, u). Therefore, $f_X(h, u) > f_X(g, w) = \beta$. Immediately, we deduce that $\sup A > \sup B$.
– If $(\psi = \nu X\psi_1, h, u) \in \textsc{Occ}(\mathfrak{E})$, then $a_{\psi,h,u} = a_{[\dddot{\psi}_1]\{\psi/X\},h,u}$.

We define the well-founded order $<$ on the set of occurrences $\textsc{Occ}(\mathfrak{E})$ as

$$(\psi, h, u) < (\psi', h', u') \overset{\text{def}}{\iff}$$
$$a_{\psi,h,u} <_\phi a_{\psi',h',u'} \text{ or } [\ a_{\psi,h,u} = a_{\psi',h',u'} \text{ and } (\psi \text{ is a strict subformula of } \psi')].$$

For contradiction, assume that there is an infinite descending chain of occurrences

$$(\psi_0, h_0, u_0) > (\psi_1, h_1, u_1) > (\psi_2, h_2, u_2) > \ldots$$

It is clear that for all $i \in \mathbb{N}$, $a_{\psi_i,h_i,u_i} \geq_\phi a_{\psi_{i+1},h_{i+1},u_{i+1}}$. Define

$$I = \{i \in \mathbb{N} \mid a_{\psi_i,h_i,u_i} >_\phi a_{\psi_{i+1},h_{i+1},u_{i+1}}\} .$$

If I is infinite, then we get an infinite $<_\phi$-descending chain of annotations, which is a contradiction. If I is finite, then there is $k \in \mathbb{N}$ such that $|I| = k$ and we get an infinite chain of formulas $\psi_{k+1}, \psi_{k+2}, \ldots$ such that for all $i \in \{k + 1, k + 2, \ldots\}$, ψ_{i+1} is a strict subformula of ψ_i. Contradiction.

Claim 59. For any occurrence $(\psi, h, u) \in \textsc{Occ}(\mathfrak{E})$, $\mathfrak{M}, h, u \models \psi$.

Proof. The proof is by induction on the well-founded order $<$ on $\textsc{Occ}(\mathfrak{E})$. Let (ψ, h, u) be an occurrence in $\textsc{Occ}(\mathfrak{E})$ and assume that the property holds for all occurrences $< (\psi, h, u)$.

- The cases $\psi = p, \neg p, P(t_1, \ldots, t_n), \neg P(t_1, \ldots, t_n)$ are easy.
- $\psi = (\psi_1 \wedge \psi_2)$. From $(\psi, h, u) \in \textsc{Occ}(\mathfrak{E})$ we get that $\mathfrak{E}_u, h \models \overline{\psi_1 \wedge \psi_2} = \overline{\psi_1} \wedge \overline{\psi_2}$. Then, $\mathfrak{E}_u, h \models \overline{\psi_1}$ and $\mathfrak{E}_u, h \models \overline{\psi_2}$. Let h_1 (h_2) be the restriction of h to $\mathsf{fvars}[\psi_1]$ ($\mathsf{fvars}[\psi_2]$). It follows that $\mathfrak{E}_u, h_1 \models \overline{\psi_1}$ and $\mathfrak{E}_u, h_2 \models \overline{\psi_2}$, which means that $(\psi_1, h_1, u), (\psi_2, h_2, u) \in \textsc{Occ}(\mathfrak{E})$. From the inductive hypothesis, we have that $\mathfrak{M}, h_1, u \models \psi_1$ and $\mathfrak{M}, h_2, u \models \psi_2$. Therefore, $\mathfrak{M}, h, u \models \psi_1$ and $\mathfrak{M}, h, u \models \psi_2$, which implies that $\mathfrak{M}, h, u \models \psi_1 \wedge \psi_2 = \psi$.
- The case $\psi = (\psi_1 \vee \psi_2)$ involves similar arguments to the ones used for the previous case.
- $\psi = \forall x\psi_1$. From $(\forall x\psi_1, h, u) \in \textsc{Occ}(\mathfrak{E})$ we get that $\mathfrak{E}_u, h \models \overline{\forall x\psi_1} = \forall x\overline{\psi_1}$. Let (r, ξ) be an arbitrary element of $\mathcal{E}$. It follows that $\mathfrak{E}_u, h \cup \{x \mapsto (r, \xi)\} \models \overline{\psi_1}$, which means that $(\psi_1, h \cup \{x \mapsto (r, \xi)\}, u)$ is in $\textsc{Occ}(\mathfrak{E})$. The inductive hypothesis gives us that $\mathfrak{M}, h \cup \{x \mapsto (r, \xi)\}, u \models \psi_1$. We deduce that $\mathfrak{M}, h, u \models \forall x\psi_1 = \psi$.
- $\psi = \exists x\psi_1$. From $(\exists x\psi_1, h, u) \in \textsc{Occ}(\mathfrak{E})$, we get that $\mathfrak{E}_u, h \models \overline{\exists x\psi_1} = \exists x\overline{\psi_1}$. There is $(r, \xi) \in \mathcal{E}$ such that $\mathfrak{E}_u, h \cup \{x \mapsto (r, \xi)\} \models \overline{\psi_1}$, which means that $(\psi_1, h \cup \{x \mapsto (r, \xi)\}, u)$ is in $\textsc{Occ}(\mathfrak{E})$. Notice that $(\psi_1, h \cup \{x \mapsto (r, \xi)\}, u) < (\exists x\psi_1, h, u)$. By the inductive hypothesis, $\mathfrak{M}, h \cup \{x \mapsto (r, \xi)\}, u \models \psi_1$ and hence $\mathfrak{M}, h, u \models \exists x\psi_1 = \psi$.
- The cases $\psi = \bigcirc\psi_1, \bullet\psi_1, \ominus\psi_1$ are similar to the next one.
- $\psi = \mu X\psi_X$. Since ψ is monodic, it has either one or none free individual variables.

- If ψ is a dom-sentence, then from $(\mu X\psi_X, \emptyset, u)$ we get that $\mathfrak{E}_u \models \overline{\mu X\psi_X}$. It follows that $\mu X\psi_X$ is in all types in $f(u).T$ and hence $[\dddot{\psi}_X]\{\mu X\psi_X/X\}$, which is also a dom-sentence, is in all types in $f(u).T$ as well, i.e. $\mathfrak{E}_u \models \overline{[\dddot{\psi}_X]\{\mu X\psi_X/X\}}$. Observe that $([\dddot{\psi}_X]\{\mu X\psi_X/X\}, \emptyset, u) < (\mu X\psi_X, \emptyset, u)$ and by the inductive hypothesis, $\mathfrak{M}, u \models [\dddot{\psi}_X]\{\mu X\psi_X/X\}$. Therefore, $\mathfrak{M}, u \models \mu X\psi_X$.
- Suppose that ψ has one free individual variable, namely x. From $(\mu X\psi_X, h, u) \in \textsc{Occ}(\mathfrak{E})$, we get that $\mathfrak{E}_u, h \models \overline{\mu X\psi_X}$. It follows that $\mu X\psi_X$ is in $r(u)$, where $h(x) = (r, \xi)$. Hence, $[\dddot{\psi}_X]\{\mu X\psi_X/X\}$ is in $r(u)$ as well. This implies that $\mathfrak{E}_u, h \models \overline{[\dddot{\psi}_X]\{\mu X\psi_X/X\}}$. Observe that $([\dddot{\psi}_X]\{\mu X\psi_X/X\}, h, u) < (\mu X\psi_X, h, u)$ and by the inductive hypothesis, $\mathfrak{M}, h, u \models [\dddot{\psi}_X]\{\mu X\psi_X/X\}$. Therefore, $\mathfrak{M}, h, u \models \mu X\psi_X$.

• $\psi = \nu X\psi_1$. We define the derivation tree for (ψ, h, u), which is a $\textsc{Occ}(\mathfrak{E})$-labeled tree. The root of the tree has label (ψ, h, u). Consider an arbitrary node n with label (χ, g, v). For every $\vdash_{\mathfrak{E},\tau}$-successor (χ', g', v') of (χ, g, v), n has a child with label (χ', g', v'). Intuitively, the derivation tree for (ψ, h, u) records all the possible $\vdash_{\mathfrak{E},\tau}$-derivations that start from (ψ, h, u). There is a derivation sequence from (ψ, h, u) to any label (χ, g, v) of a node of the tree and hence $a_{\chi,g,v} \leq_\phi a_{\psi,h,u}$. We prune the derivation tree at the nodes where the inductive hypothesis can be applied to the label. So, for a leaf n of the pruned derivation tree with label (χ, g, v), we have that $(\chi, g, v) < (\psi, h, u)$. Observe that the case of (χ, g, v) not having a $\vdash_{\mathfrak{E},\tau}$-successor, i.e. when χ is an atomic fp-sentence or the negation of an atomic fp-sentence, is covered by the inductive hypothesis. See Figure 6 for an example of a pruned derivation tree. Notice that the tree is pruned as soon as a sub-fp-sentence of $\nu X\psi_X$ is encountered. This implies that all fp-sentences that appear in non-leaf nodes have $\nu X\psi_X$ as sub-fp-sentence.

We will say that an occurrence (χ, g, v) is true under a first-order temporal structure $\mathfrak{M}'$ if $\mathfrak{M}', g, u \models \psi$. It is false under $\mathfrak{M}'$ if it is not true under $\mathfrak{M}'$. We will also say that a node of the derivation is true (false) under $\mathfrak{M}'$ if its label is true (false). Consider an arbitrary non-leaf node n of the tree and observe that if all its children are true under $\mathfrak{M}'$, then n is true under $\mathfrak{M}'$. Equivalently, if n is false under $\mathfrak{M}'$, then at least one child of n is false under $\mathfrak{M}'$. Let us note that all the leaf nodes of the pruned derivation tree for (ψ, h, u) are true under $\mathfrak{M}$.

The proof proceeds by contradiction. Assume that the root of the derivation tree is false under $\mathfrak{M}$, i.e. $\mathfrak{M}, h, u \not\models \psi$. Then, the unique child n of the root is also false under $\mathfrak{M}$. That is, $\mathfrak{M}, h, u \not\models [\dddot{\psi}_1]\{\psi/X\}$. Notice that n is not a leaf node, because then it would be true under $\mathfrak{M}$. We can continue like this and construct an infinite sequence of labels following an infinite branch of the pruned derivation tree, in which all labels are false under $\mathfrak{M}$. Note that in the case of a conjunction or a universally quantified formula, if more than one children are false under $\mathfrak{M}$, we choose one with the $<_\phi$-least ν_ϕ-signature under $\mathfrak{M}$. Observe that there is no least fixpoint sentence in this sequence, because if there was one, say $(\mu X\psi_X, g, v)$, then the next occurrence would be $([\dddot{\psi}_X]\{\mu X\psi_X/X\}, g, v)$ and we could apply the inductive hypothesis. But, in order to have an infinite derivation sequence we must be going infinitely often through fixpoint fp-sentences or else the sequence would terminate at some atomic fp-sentence or the negation of an atomic fp-sentence. Consider the set of greatest fixpoint fp-sentences that appear infinitely often in s (there may be more than one) $\{\nu X_1\psi_{X_1}, \ldots, \nu X_k\psi_{X_k}\}$ and pick the one, say $\nu Y\psi_Y$, that binds the $<$-least fixpoint variable, i.e. $Y < X_1, \ldots, Y < X_k$. There may be more greatest fixpoint fp-sentences that appear in s, but they will stop appearing from a point on. After that point, find the first appearance of $\nu Y\psi_Y$ in s. Starting from there, every time we go through $\nu Y\psi_Y$, ν_ϕ-signature strictly decreases (this can be shown will similar arguments to the ones used in Theorem 52 having Proposition 33 in mind) and hence we get an infinite $<_\phi$-descending chain of ν_ϕ-signatures, which is a contradiction. □

From $(\phi, \emptyset, w) \in \textsc{Occ}(\mathfrak{E})$ and the above claim, we get that $\mathfrak{M}, w \models \phi$ and hence ϕ is satisfiable.
□

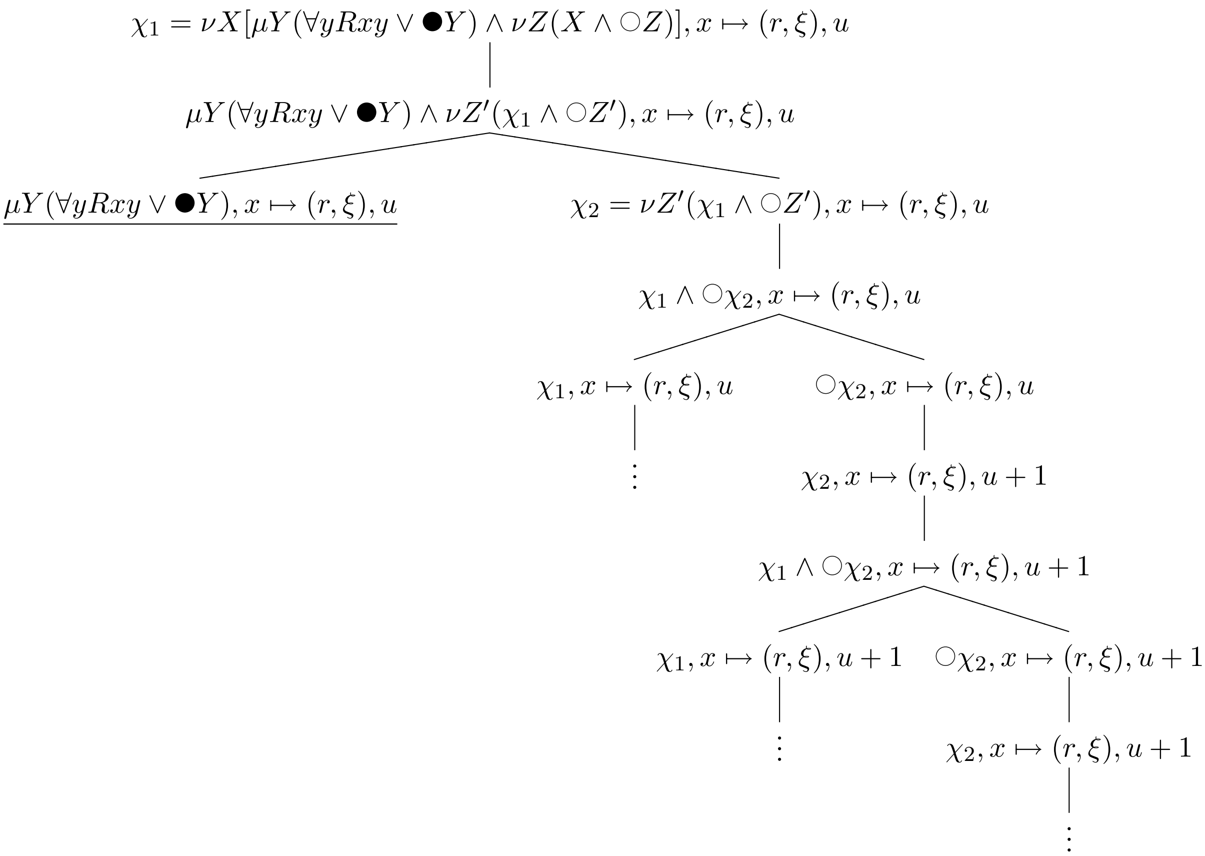

FIGURE 6. Pruned derivation tree.

6.5. **Finite satisfiability & satisfaction in finitary quasimodels.** First of all, let us observe that finite domains give rise to different logics. Let $\Box$ mean 'at every moment' and H mean 'always in the past'. Easily, we see that both these temporal connectives can be expressed with ν, $\bigcirc$, and $\ominus$.

$$\mathsf{H}\phi \equiv \nu X(\ominus\phi \wedge \ominus X) \qquad\qquad \Box\phi \equiv= \underbrace{\nu X(\ominus\phi \wedge \ominus X)}_{\mathsf{H}\phi} \vee \phi \vee \underbrace{\nu X(\bigcirc\phi \wedge \bigcirc X)}_{\mathsf{G}\phi}$$

Consider the formula $\phi = \Box\exists x(Qx \wedge \mathsf{H}\neg Qx)$ (the example is taken from [32]). We verify that ϕ is satisfied in the temporal structure $\mathfrak{M}$ with domain $\mathbb{N}$ that interprets Q as $Q^{I_u} = \{u\}$. It is not, however, finitely satisfiable.

Theorem 60. Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence. Then, ϕ is finitely satisfiable if and only if there is a well-founded adorned finitary quasimodel for ϕ, in which ϕ is satisfied.

Proof. Fix a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence ϕ. The $\Rightarrow$ direction is easy. The construction of Theorem 52 yields a well-founded adorned finitary quasimodel. The $\Leftarrow$ direction requires a minor modification of the argument. Suppose that ϕ is satisfied in some well-founded adorned finitary quasimodel $\mathfrak{m} = (f, \mathscr{R}, \tau)$ for ϕ. There is a family of finite σ_{surr}-structures $\{\mathfrak{D}_u = (\mathcal{D}_u, \cdot^{\mathfrak{D}_u})\}_{u\in\mathbb{N}}$ such that $\mathfrak{D}_u$ realizes $f(u)$. Since there are finitely many realizable state candidates, we can assume without loss of generality that the quasistates are realized by finitely many σ_{surr}-structures. It follows that the set $\{|\mathfrak{D}_{u[t]}| \mid u \in \mathbb{N}, t \in f(u)\}$ is finite and hence it has a maximum element, namely M. We define the finite set

$$\mathcal{E} = \mathscr{R} \times M = \{(r, \xi) \mid r \in \mathscr{R} \text{ and } \xi \in M\} .$$

Observe that, for any $u \in \mathbb{N}$, we can define a left-total, injective, and right-total relation $R_u \subseteq \mathcal{D} \times \mathcal{E}$ and a σ_{surr}-structure $\mathfrak{E}_u = (\mathcal{E}, \cdot^{\mathfrak{E}_u})$ like in Theorem 56 so that $\mathfrak{E}_u$ realizes $f(u)$. We can arrange that the constants are interpreted rigidly. The proof from this point on proceeds exactly as in Theorem 56. ☐

7. Complexity

Consider a formula ϕ of propositional temporal logic with no fixpoint operators. Employing combinatorial arguments, a pre-model for ϕ can be chopped down to an ultimately periodic one, which can be viewed as a finite object. This is the approach taken in [49]. So, the existence of a pre-model in which ϕ is satisfied can be checked with a non-deterministic algorithm that guesses the types for an ultimately periodic pre-model and checks that the temporal constraints are satisfied. Alternatively, the pre-model can be viewed as an ω-word over the set of all types for ϕ. Then, a Büchi automaton can be constructed that accepts exactly those pre-models in which ϕ is satisfied [59]. Thus, satisfiability is reduced to nonemptiness of a Büchi automaton. The latter approach extends elegantly to the case of propositional temporal logic with fixpoint operators [58]. The authors in [32] extend the technique of [49] to quasimodels and show how a quasimodel can be chopped down to an appropriate ultimately periodic one.

We have opted here for the automata-theoretic approach, since it abstracts away much combinatorial thinking. Both for the arbitrary domain case and for the finite domain case the idea is the same: Given a sentence ϕ, we describe an automaton A_ϕ that takes as input an infinite sequence of state candidates for ϕ and accepts if and only if this sequence can be extended to a well-founded (finitary) quasimodel in which ϕ is satisfied. Checking (finite) satisfiability for ϕ is then reduced to checking nonemptiness for A_ϕ. We note that the algorithm for the first-order part is plugged in the algorithm for nonemptiness.

7.1. Arbitrary domain.

Arbitrary domain. Vardi shows in [58] that, given a propositional temporal sentence ϕ with fixpoints, a Büchi automaton can be constructed that accepts exactly the well-founded adorned pre-models in which ϕ is satisfied. Moreover, the number of states of this automaton is singly exponential in the size of ϕ. This result extends in our case for sequences of types. That is, we can construct a 'Vardi automaton' (call it A_r) that accepts exactly those sequences of types that are temporally consistent and well-founded (potentially well-founded, to be more accurate).

We want to construct an automaton A_ϕ that accepts sequences of state candidates that can be made into appropriate quasimodels for ϕ. Any information about runs is missing from such sequences. So, the difficult part is checking that each type at each moment is hit be a well-founded run. It seems appropriate to use an alternating automaton that launches a new process for each type at each moment in order to perform exhaustively this check. It turns out, however, that we need more information than what is available from the state candidates. For that reason, we "decorate" each type at each moment with a subset of the states of A_r. A state q is in the decoration of a type $t \in f(u).T$ if there is a finite sequence in $\prod_{v\in[0,u]} f(v).T$ and a computation of A_r on that sequence that ends at state q. Informally, we can think of these decorations as "paused simulations" of computations of A_r. Any possible computation of A_r on any possible finite sequence of types in f is considered. This information would then enable the alternating automaton to function correctly. It guesses a paused simulation, i.e. a state of A_r from the decoration, and resumes the simulation of A_r guessing at each step the type to follow. We also define another automaton that takes as input a decorated sequence of state candidates and performs all the rest of the necessary checks: whether the decoration is as it should, satisfaction of ϕ, realizability, etc.

Fix a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}$-sentence ϕ. Recall that $\mathsf{Types}(\phi)$ is the set of types for ϕ, $\mathsf{StateC}(\phi)$ is the set of states candidates for ϕ, and $\mathsf{Real}(\phi)$ is the set of realizable state candidates for ϕ. The size of $\mathsf{Types}(\phi)$ is exponential in the size of ϕ and the sizes of $\mathsf{StateC}(\phi)$ and $\mathsf{Real}(\phi)$ are doubly exponential in the size of ϕ.

From Theorem 52 and Theorem 56, we have that ϕ is satisfiable if and only if there is an infinite word $f \in \mathsf{StateC}(\phi)^\omega$ such that the following hold.

- For every $u \in \mathbb{N}$, $f(u)$ is a realizable state candidate (hence f is a state function for ϕ).
- There is $w \in \mathbb{N}$ such that ϕ is in all types in $f(w).T$
- Every type at every moment is hit by a well-founded run.
- For any $c \in \mathsf{con}[\phi]$, the function r_c, defined as $r_c(u) = f(u).T^{con}(c)$, is a well-founded run.

We will construct a Büchi automaton A that accepts the ω-language

$$L(A) = \{f \in \mathsf{StateC}(\phi)^\omega \mid f \text{ can be extended to a well-founded quasimodel that satisfies } \phi\} .$$

Clearly, ϕ is satisfiable if and only if $L(A)$ is not empty.

Definition 61 (suitability, type-sequence, consistency, well-foundedness, potential well-foundedness)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}$-sentence and t_1, t_2 be types for ϕ.

- We say that the pair (t_1, t_2) is *suitable* if for any ψ,

$$\bigcirc\psi \in t_1 \implies \psi \in t_2 \qquad \bullet\psi \in t_2 \implies \psi \in t_1 \qquad \ominus\psi \in t_2 \implies \psi \in t_1 .$$

- An infinite word r over $\mathsf{Types}(\phi)$ is a *type-sequence* for ϕ.
- We say that a type-sequence r is *consistent* if $r(0)$ contains all $\ominus$-fp-sentences in $\mathsf{cl}_1[\phi]$ and for all $u \in \mathbb{N}$, the pair $\langle r(u), r(u+1)\rangle$ is suitable.
- Let τ be a choice function for a consistent type-sequence r. The pair (r, τ) is a *consistent adorned type-sequence* for ϕ.
- We say that a consistent adorned type-sequence (r, τ) is a *well-founded* if the regeneration relations for all least fixpoint fp-sentences in $\mathsf{cl}_1[\phi]$ are converse well-founded.
- We say that a consistent type-sequence r is *potentially well-founded* if there is a choice function τ such that (r, τ) is a well-founded consistent adorned type-sequence for ϕ.

Let us construct an automaton $A_r = (Q_r, \mathsf{Types}(\phi), q_r^0, \delta_r, F_r)$ that accepts potentially well-founded consistent type-sequences for ϕ, i.e the the ω-language

$$L(A_r) = \{r \in \mathsf{Types}(\phi)^\omega \mid r \text{ is a potentially well-founded consistent type-sequence for } \phi\} .$$

Consider the Büchi automaton A'_r that accepts well-founded consistent adorned type-sequences for ϕ. Let t be a type for ϕ. Define T_t to be the set of functions that map each disjunction $(\psi_1 \vee \psi_2)$ in t to a disjunct ψ_i that is in t, and $T = \bigcup_{t \in \mathsf{Types}(\phi)} T_t$. It is obvious that an adorned type-sequence can be represented as an infinite word ρ over $\mathsf{Types}(\phi) \times T$, where $\rho(u).\tau$ is a choice function for $\rho(u).t$ for all $u \in \mathbb{N}$.

$$L(A'_r) = \{\rho \in (\mathsf{Types}(\phi) \times T)^\omega \mid \rho \text{ is a well-founded consistent adorned type-sequence for } \phi\}$$

From results of Vardi in [58], we have that the number of states of A'_r is exponential in the size of ϕ. Since the size of the alphabet is at most

$$\flat(\phi) \times \underbrace{2 \times 2 \times \cdots \times 2}_{|\mathsf{cl}_1[\phi]| \text{ times}} = \flat(\phi) \times 2^{|\mathsf{cl}_1[\phi]|}$$

and hence exponential in the size of ϕ, the entire description of A'_r needs only exponential space. Its construction is effective and the algorithm runs in $\mathsf{EXPSPACE}$. A_r accepts the projection of $L(A'_r)$ on the alphabet $\mathsf{Types}(\phi)$ simply by guessing the choice functions.

Let f be a state function for ϕ, u be a moment, and t be a type in $f(u).T$. We want to check whether there exists a function r in $\prod_{v\in\mathbb{N}} f(v).T$ that goes throught t at u, i.e. $r(u) = t$, and that is consistent and potentially well-founded. If we could somehow guess this function r, then we could have the automaton A_r compute on r and decide whether r is consistent and potentially well-founded. Consider the set of all possible finite prefixes $\prod_{v\in[0,u]} f(v).T$ and the set of all possible infinite suffixes $\prod_{v\in[u+1,\infty)} f(v).T$.

- Suppose that there is a consistent and potentially well-founded r in $\prod_{v\in\mathbb{N}} f(v).T$ such that $r(u) = t$. Then, there is an accepting computation of A_r on r.

$$q_r^0 \xrightarrow{r(0)} p_r^0 \xrightarrow{r(1)} \cdots \xrightarrow{r(u-1)} p_r^{u-1} \xrightarrow{r(u)=t} \mathbf{p}_\mathbf{r}^\mathbf{u} \xrightarrow{r(u+1)} p_r^{u+1} \xrightarrow{r(u+2)} \cdots$$

That is, final states appear infinitely often in the sequence of states $q_r^0, p_r^0, p_r^1, p_r^2, \ldots$ Equivalently, final states appear infinitely often in the sequence $p_r^{u+1}, p_r^{u+2}, p_r^{u+3}, \ldots$ Extend the transition function δ_r of A_r to $\hat{\delta}_r : Q_r \times \mathsf{Types}(\phi)^\star \to \wp(Q_r)$ so that $\hat{\delta}_r(q, r)$ gives the set of states the automaton A_r could be in after computing on the finite word r. An inductive definition for $\hat{\delta}_r$ is the following.

$$\hat{\delta}_r(q,\varepsilon) = \{q\} \qquad\qquad \hat{\delta}_r(q,\rho t) = \bigcup_{p\in\hat{\delta}_r(q,\rho)} \delta_r(p,t)$$

It is clear that $p_r^u \in \hat{\delta}_r(q_r^0, r(0)r(1)\dots r(u))$.

- Consider all the sequences in $\prod_{v\in[0,u]} f(v).T$ that end at $t \in f(u).T$. The set of states the automaton A_r could be in after computing on any of these finite sequences is

$$\pi(u,t) = \left\{\hat{\delta}_r(q_r^0,\rho) \mid \rho \in \prod_{v\in[0,u]} f(v).T \text{ and } \rho(u) = t\right\} .$$

Suppose that there is a state $p_r^u \in \pi(u,t)$ and a computation of A_r on an infinite sequence r_2 in $\prod_{v\in[u+1,\infty)} f(v).T$ starting from p_r^u that is accepting. There is a finite sequence r_1 in $\prod_{v\in[0,u]} f(v).T$ such that there is a computation of A_r, starting from its initial state, that ends at state p_r^u. The sequence $r = r_1 \cup r_2$ is then a potentially well-founded consistent run in f.

The above discussion shows that if we know at any moment u and at any type $t \in f(u).T$ the set of states $\pi(u,t)$, then we can guess a state q in $\pi(u,t)$ and start a computation of A_r from q guessing at each moment the next type A_r should read. We verify that the computation is accepting and claim that there is a potentially well-founded consistent run in f that goes throught t at u. So, we "decorate" a type t in a quasistate $f(u)$ with a set $f(u).\pi(t) \subseteq Q_r$ that is equal to $\pi(u,t)$. The decoration can be defined inductively as follows.

$$f(0).\pi(t) = \delta_r(q_r^0,t) \qquad\qquad f(u+1).\pi(t) = \bigcup_{t'\in f(u).T}\left\{\bigcup_{q\in f(u).\pi(t')} \delta_r(q,t)\right\} \tag{1}$$

For a state function f, equation (1) defines a unique decoration.

Definition 62 (decorated state candidate, decorated state function)**.** Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}$-sentence and $A = (Q, \mathsf{Types}(\phi), q_0, \delta, F)$ be a Büchi automaton.

- An A-*decorated state candidate* for ϕ is a triple (T, T^{con}, π), where (T, T^{con}) is a state candidate for ϕ and $\pi : T \to \wp(Q)$.
- An A-decorated state candidate (T, T^{con}, π) is called *realizable* if (T, T^{con}) is realizable.
- An A-*decorated state function* for ϕ is a function f that maps each moment $u \in \mathbb{N}$ to an A-decorated realizable state candidate $f(u) = (T_u, T_u^{con}, \pi_u)$. We will write $T_u = f(u).T$, $T^{con} = f(u).T^{con}$, and $\pi_u = f(u).\pi$.
- We say that ϕ is *satisfied* in an A-decorated state function f if there is $u \in \mathbb{N}$ such that ϕ is in all types in $f(u).T$.
- We say that the an A-decorated state function f is *correctly decorated* if its decoration satisfies equation (1) (put δ where δ_r).

Let $\mathsf{DStateC}(\phi)$ be the set of all A_r-decorated state candidates. Observe that there are at most

$$\begin{aligned}\natural(\phi) &= 2^{\flat(\phi)} \times \underbrace{\flat(\phi)\times\flat(\phi)\times\cdots\times\flat(\phi)}_{|\mathsf{con}[\phi]| \text{ times}} \times \underbrace{2^{|Q_r|}\times 2^{|Q_r|}\times\cdots\times 2^{|Q_r|}}_{\flat(\phi) \text{ times}} \\ &= 2^{\flat(\phi)} \times \flat(\phi)^{|\mathsf{con}[\phi]|} \times 2^{|Q_r|\times\flat(\phi)}\end{aligned}$$

of them. That is, $|\mathsf{DStateC}(\phi)|$ is doubly exponential in the size of ϕ. We construct now a Büchi automaton A_1 that accepts the ω-language

$$L(A_1) = \{f \in \mathsf{DStateC}(\phi)^\omega \mid f \text{ is a correctly } A_r\text{-decorated state function that satisfies } \phi\} .$$

A_1 needs to remember the entire previous A_r-decorated state candidate in order to check the correctness of the decoration. Satisfaction of ϕ is checked by guessing the moment where ϕ is satisfied. The set of states of A_1 is defined as

$$Q_1 = \{q_0, q_F\} \cup \left(\{q_1, q_\phi, q_2\} \times \mathsf{DStateC}(\phi)\right) .$$

q_0 is the initial state, q_F is the fail state, q_1 signifies that we should be checking just for realizability and decoration correctness, q_ϕ that we should be checking for realizability, decoration correctness and verify that ϕ appears in all types of the current state candidate, and q_2 that we have already verified the satisfaction of ϕ and that from now on we will only be checking realizability and decoration correctness. It is clear that $|Q_1|$ is doubly exponential in the size of ϕ. The transition function for A_1 is defined as

$$\delta_1(q_0, \mathfrak{C}) = \begin{cases} \{(q_1, \mathfrak{C}), (q_\phi, \mathfrak{C})\}, & \text{if } \mathfrak{C} \text{ is realizable and its decoration correct} \\ \{q_F\}, & \text{otherwise} \end{cases}$$
$$\delta_1(q_F, \mathfrak{C}) = \{q_F\}$$
$$\delta_1(\langle q_1, \mathfrak{C}\rangle, \mathfrak{C}') = \begin{cases} \{(q_1, \mathfrak{C}'), (q_\phi, \mathfrak{C}')\}, & \text{if } \mathfrak{C}' \text{ is realizable and its decoration correct} \\ \{q_F\}, & \text{otherwise} \end{cases}$$
$$\delta_1(\langle q_\phi, \mathfrak{C}\rangle, \mathfrak{C}') = \begin{cases} \{(q_2, \mathfrak{C}')\}, & \text{if } \mathfrak{C}' \text{ is realizable, its decoration correct, and it satisfies } \phi \\ \{q_F\}, & \text{otherwise} \end{cases}$$
$$\delta_1(\langle q_2, \mathfrak{C}\rangle, \mathfrak{C}') = \begin{cases} \{(q_2, \mathfrak{C}')\}, & \text{if } \mathfrak{C}' \text{ is realizable and its decoration correct} \\ \{q_F\}, & \text{otherwise} \end{cases}$$

The set of final states is $F_1 = \{q_2\} \times \mathsf{DStateC}(\phi)$.

Supposing that we are given a correctly A_r-decorated state function f, we can easily check whether it can be extended to a well-founded quasimodel with an alternating Büchi automaton. The idea is that at every moment u and at every type $t \in f(u).T$ we pick an A_r-state from the decoration $f(u).\pi(t)$ and launch a new process that continues the simulation of A_r on some run that it guesses nondeterministically. Note that for each constant $c \in \mathsf{con}[\phi]$, we have to verify that the sequence r_c, defined as $r_c(u) = f(u).T^{con}(c)$, is a potentially well-founded run. We define the alternating Büchi automaton $A_2 = (Q_2, \mathsf{DStateC}(\phi), q_0, \delta_2, F_2)$, which accepts the ω-language $L(A_2)$, for which it holds that for any $f \in L(A_1)$,

$$f \in L(A_2) \iff f \text{ can be extended to a well-founded quasimodel .}$$

We have to introduce states for the "main" process that launches all the rest, for the processes that search for potentially well-founded runs, and for the processes that check the runs for the constants. So, we define

$$Q_2 = \{q_0, q_1\} \cup Q_r \cup (\mathsf{con}[\phi] \times Q_r) \ .$$

The main process involves states q_0 and q_1. The transition function for these states is defined as

$$\delta_2(q_0, \mathfrak{C}) = \bigwedge_{c \in \mathsf{con}[\phi]} \left\{ \bigvee_{\substack{q_r \in \mathfrak{C}.\pi(t_c) \\ t_c = \mathfrak{C}.T^{con}(c)}} (c, q_r) \right\} \wedge \bigwedge_{t \in \mathfrak{C}.T} \left\{ \bigvee_{q_r \in \mathfrak{C}.\pi(t)} q_r \right\} \wedge q_1$$
$$\delta_2(q_1, \mathfrak{C}) = \bigwedge_{t \in \mathfrak{C}.T} \left\{ \bigvee_{q_r \in \mathfrak{C}.\pi(t)} q_r \right\} \wedge q_1 \ .$$

Observe that the processes for constants are launched from the initial state q_0. The processes for the types are launched both from q_0 and q_1. Consider now a process for a type. When presented with the next A_r-decorated state candidate, it guesses the type it should continue on and also guesses the next A_r-state from the transition function δ_r (remember that A_r is nondeterministic). That is,

$$\delta_2(q_r, \mathfrak{C}) = \bigvee_{t \in \mathfrak{C}.T} \bigvee_{q_r' \in \delta_r(q_r, t)} q_r' \ .$$

A process for a constant c does not have to guess the next type. It reads the next type from $\mathfrak{C}.T^{con}(c)$. So, the transition function is defined as

$$\delta_2(\langle c, q_r\rangle, \mathfrak{C}) = \bigvee_{q'_r \in \delta_r(q_r, \mathfrak{C}.T^{con}(c))} (c, q'_r) .$$

Notice that we need not check the suitability of the next type. This is taken care of by the automaton A_r, which checks for temporal consistency as well as for potential well-foundedness. The set of final states is $F_2 = F_r \cup \{q_1\}$, i.e. the final states of the automaton A_r that each process simulates and the state q_1 so that the branch of the main process is accepting as well.

Let us see now why $L(A_2)$ satisfies the desired property. Let f be in $L(A_1)$. That is, f is a correctly A_r-decorated state function that satisfies ϕ. Suppose that f is also in $L(A_2)$. Consider the computation tree of the alternating automaton A_2 on f. Take a moment u and a type t in $f(u).T$. The process that starts from this type corresponds to an infinite branch of the computation tree. This branch is a computation of A_r starting from some state p_r^u in $f(u).\pi(t)$.

$$p_r^u \xrightarrow{t^{u+1}\in f(u+1).T} p_r^{u+1} \xrightarrow{t^{u+2}\in f(u+2).T} p_r^{u+2} \xrightarrow{t^{u+3}\in f(u+3).T} p_r^{u+3} \xrightarrow{t^{u+4}\in f(u+4).T} \cdots$$

The types $t^{u+1}, t^{u+2}, t^{u+3}, \ldots$ are selected nondeterministically and the computation is accepting, i.e. there is at least one accepting state in F_r that appears infinitely often. Since p_r^u is in $f(u).\pi(t)$ and the decoration has been verified to be correct, we get that there is a sequence of types $t^0, t^1, \ldots, t^u = t$ with $t^v \in f(v).T$ for all $v \in \{0, \ldots, u\}$, such that the automaton A_r computes as follows.

$$q_r^0 \xrightarrow{t^0\in f(0).T} p_r^0 \in f(0).\pi(t^0) \xrightarrow{t^1\in f(1).T} p_r^1 \in f(1).\pi(t^1) \xrightarrow{t^2\in f(2).T} \cdots$$

$$\cdots \xrightarrow{t^{u-1}\in f(u-1).T} p_r^{u-1} \in f(u-1).\pi(t^{u-1}) \xrightarrow{t^u\in f(u).T} p_r^u \in f(u).\pi(t^u)$$

Immediately, we see that the sequence $r = t^0, t^1, t^2, \ldots$ is a type-sequence that is accepted by A_r and hence it is a potentially well-founded run in f that goes through type t at u. So, there is a choice function τ_r for r such that (r, τ_r) is well-founded. Similarly, we argue for the runs for the constants. Take the set $\mathscr{R}$ of these runs for each type and each moment and also the runs for the constants. The choice function τ for the entire quasimodel is defined in the obvious way from $\{\tau_r\}_{r\in\mathscr{R}}$. Clearly, $(f, \mathscr{R}, \tau)$ is a well-founded quasimodel for ϕ. For the converse, we assume that there is a well-founded quasimodel $(f, \mathscr{R}, \tau)$ based on f. Then, each process makes the necessary nondeterministic choices so that it simulates an accepting computation of A_r.

Observe that $|Q_2|$ is exponential in the size of ϕ. From A_2, we can construct a language-equivalent nondeterministic automaton A'_2 with an exponential blow-up in the number of states [44]. That is, the number of states of A'_2 is doubly exponential in the size of ϕ. With a simple quadratic construction [9], we get the automaton A' that accepts the language

$$\begin{aligned} &L(A') = L(A_1) \cap L(A'_2) = L(A_1) \cap L(A_2) = \\ &\qquad \{f \in \mathsf{DStateC}(\phi)^\omega \mid f \text{ can be extended to a well-founded quasimodel that satisfies } \phi\} . \end{aligned}$$

The projection of $L(A')$ on $\mathsf{StateC}(\phi)$ is accepted by the automaton A, which guesses the decorations. The number of states of A is doubly exponential in the size of ϕ.

Theorem 63. Let $\mathcal{F}$ be a sublanguage of $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}$.

- Suppose that there is an algorithm which, given a state candidate $\mathfrak{C}$ for a $\mathcal{F}$-sentence, can decide whether $\mathfrak{C}$ is realizable using exponential space in the size of ϕ. Then, the satisfiability problem for $\mathcal{F}$-sentences is in $\mathsf{EXPSPACE}$.
- Suppose that there is an algorithm which, given a state candidate $\mathfrak{C}$ for a $\mathcal{F}$-sentence, can decide whether $\mathfrak{C}$ is realizable using doubly exponential time in the size of ϕ. Then, the satisfiability problem for $\mathcal{F}$-sentences is in $\mathsf{2EXPTIME}$.

Proof. We present a nondeterministic algorithm. Fix a $\mathcal{F}$-sentence ϕ. Within exponential space we construct the automaton A_r, that accepts potentially well-founded consistent type-sequences for ϕ. Consider the automaton A over the alphabet $\mathsf{StateC}(\phi)$ defined as in the preceding discussion. ϕ is satisfiable if and only if $L(A)$ is nonempty. We check non-emptiness in the usual way: find a path from the initial state q_0 of A to some accepting state q_Y and then a path from q_Y back to q_Y. A is too big to be constructed within exponential space, so we construct parts of it on-the-fly. We find the next state in a path by guessing some state and then verifying that it is indeed a successor of the current state. This check involves guessing a state candidate and running the algorithm for realizability. If checking for realizability requires exponential space, then the described algorithm is in $\mathsf{NEXPSPACE}$, which is equal to $\mathsf{EXPSPACE}$ [48]. If checking for realizability requires doubly exponential time, then observe that a nondeterministic algorithm is an alternating algorithm. Alternating exponential space is equal to doubly exponential time, i.e. $\mathsf{AEXPSPACE} = \mathsf{2EXPTIME}$ [8]. Hence, the described algorithm is in $\mathsf{2EXPTIME}$. $\square$

7.2. **Finite Domain.** The finite domain case is significantly more involved than the arbitrary domain case. First, we show a "quasimodel-theoretic" proposition that reveals the defining characteristics of a well-founded finitary quasimodel. It immediately becomes obvious that we have to extend the decoration technique used in the previous section, because the "processes" that perform the checks on runs need more information to function correctly.

Suppose that ϕ is satisfied in some well-founded finitary quasimodel $\mathfrak{m} = (f, \mathscr{R}, \tau)$. Every quasistate $f(u)$ is realized by some finite σ_{surr}-structure and $\mathscr{R}$ is finite.

Remember that the Büchi automaton A_r, as defined in Section 7.1, accepts potentially well-founded consistent type-sequences for ϕ. Fix a run $r \in \mathscr{R}$. Clearly, r is potentially well-founded and hence there is a computation of A_r on r

$$q_r^0 \xrightarrow{r(0)} p_0 \xrightarrow{r(1)} p_1 \xrightarrow{r(2)} p_2 \xrightarrow{r(3)} p_3 \xrightarrow{r(4)} \dots$$

that is accepting, i.e. accepting states appear infinitely often. Let s_r be this accepting computation on r (starting from p_0) and define

$$\rho_r : \mathbb{N} \to \mathsf{Types}(\phi) \times Q_r \qquad\qquad \rho_r(u) = \langle r(u), s_r(u) \rangle$$

We will write $\rho_r(u).t = r(u)$ and $\rho_r(u).q = s_r(u)$. We also define the function g which maps each moment u to the set $\{\langle r, r(u), \rho_r(u)\rangle \mid r \in \mathscr{R}\}$. That is, g associates to each moment u a function that maps each run $r \in \mathscr{R}$ to the pair $\langle r(u), s_r(u)\rangle$. Observe that $g(u) = g(v)$ implies that

$$f(u).T = \{r(u) \mid r \in \mathscr{R}\} = \{g(u)(r).t \mid r \in \mathscr{R}\} = \{g(v)(r).t \mid r \in \mathscr{R}\} = \{r(v) \mid r \in \mathscr{R}\} = f(v).T$$
$$f(u).T^{con}(c) = r_c(u) = g(u)(r_c).t = g(v)(r_c).t = r_c(v) = f(v).T^{con}(c) \text{ , for all } c \in \mathsf{con}[\phi]$$

and hence $f(u) = f(v)$. Since there are finitely many runs in $\mathscr{R}$, g can take only finitely many values.

By satisfaction of ϕ, we get that there is a moment w such that ϕ is in all types in $f(w).T$. After w, we can wait until all values of g that appear finitely many times have finished appearing. Call this moment I. This means that every value $g(u)$, for $u \geq I$, appears infinitely many times in the interval $[I, \infty)$. Since there are only finitely many runs, we can wait until a final state appears in the computations $\{s_r\}_{r \in \mathscr{R}}$ of A_r on all runs and then wait a bit longer until the value $g(I)$ reappears. Call this moment J. See Figure 7. The following hold.

(1) For every moment $u \in [0, J-1]$ and every type $t \in f(u).T$, there is a finite sequence r in $\prod_{v \in [0,J-1]} f(v).T$ such that $r(u) = t$ and there is a computation

$$q_r^0 \xrightarrow{r(0)} p_0 \xrightarrow{r(1)} \dots \xrightarrow{r(I)} p_I \xrightarrow{r(I+1)} p_{I+1} \xrightarrow{r(I+2)} \dots \xrightarrow{r(J-1)} p_{J-1}$$

of A_r on r such that a final state is in $\{p_I, \dots, p_{J-1}\}$ and $p_{J-1} \xrightarrow{r(I)} p_I$.

(2) For every constant $c \in \mathsf{con}[\phi]$, there is a computation

$$q_r^0 \xrightarrow{r_c(0)} p_0 \xrightarrow{r_c(1)} \dots \xrightarrow{r_c(I)} p_I \xrightarrow{r_c(I+1)} p_{I+1} \xrightarrow{r_c(I+2)} \dots \xrightarrow{r_c(J-1)} p_{J-1}$$

of A_r on r_c such that a final state is in $\{p_I, \dots, p_{J-1}\}$ and $p_{J-1} \xrightarrow{r_c(I)} p_I$.

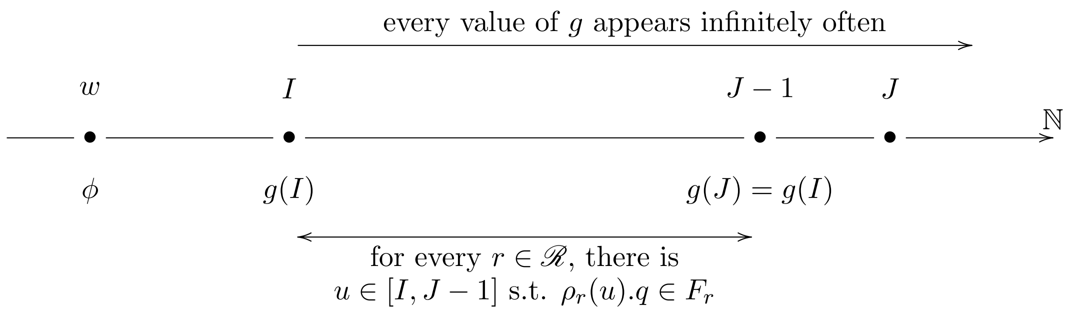

FIGURE 7. Satisfaction of ϕ in a well-founded finitary quasimodel.

Let us prove the first claim. Fix a moment $u \in [0, J-1]$ and a type $t \in f(u).T$. There is a run $r \in \mathscr{R}$ such that $r(u) = t$. Consider the sequence $\{\rho_r(v) = \langle r(v), s_r(v)\rangle\}_{v\in\mathbb{N}}$, which describes the following computation of A_r on r.

$$q_r^0 \xrightarrow{r(0)} s_r(0) \xrightarrow{r(1)} s_r(1) \xrightarrow{r(2)} \dots \xrightarrow{r(I)} s_r(I) \xrightarrow{r(I+1)} s_r(I+1) \xrightarrow{r(I+2)} \dots$$
$$\xrightarrow{r(J-1)} s_r(J-1) \xrightarrow{r(J)} s_r(J) \xrightarrow{r(J+1)} \dots$$

Observe that there is a final state in $s_r(I), \dots, s_r(J-1)$. From $g(I) = g(J)$, it follows that $r(I) = r(J)$ and $s_r(J) = s_r(I)$. So, we have that

$$s_r(J-1) \xrightarrow{r(J)} s_r(J) \implies s_r(J-1) \xrightarrow{r(I)} s_r(I) .$$

The restriction of r to $[0, J-1]$ satisfies all the conditions. With similar arguments we show the second claim.

Proposition 64. Let ϕ be a $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}$-sentence. Then, ϕ is satisfied in a well-founded finitary quasimodel if and only if there exist two finite sequences f_1, f_2 of realizable state candidates for ϕ of length I and $(J-I)$ respectively such that the following hold (define $(f_1 \star f_2)$ as the concatenation of f_1 and f_2).

(1) There is a moment $w \in [0, I)$ such that ϕ is in all types in $f_1(w).T$.
(2) For every moment $u \in [0, J-1]$ and every type $t \in (f_1 \star f_2)(u).T$, there is finite sequence r in $\prod_{v\in[0,J-1]}(f_1 \star f_2)(v).T$ and a computation
$$q_r^0 \xrightarrow{r(0)} p_0 \xrightarrow{r(1)} p_1 \xrightarrow{r(2)} \dots \xrightarrow{r(I)} p_I \xrightarrow{r(I+1)} p_{I+1} \xrightarrow{r(I+2)} \dots \xrightarrow{r(J-1)} p_{J-1}$$
of A_r on r such that a final state is in $\{p_I, \dots, p_{J-1}\}$ and $p_{J-1} \xrightarrow{r(I)} p_I$.
(3) For every constant $c \in \mathsf{con}[\phi]$, there is a computation
$$q_r^0 \xrightarrow{r_c(0)} p_0 \xrightarrow{r_c(1)} p_1 \xrightarrow{r_c(2)} \dots \xrightarrow{r_c(I)} p_I \xrightarrow{r_c(I+1)} p_{I+1} \xrightarrow{r_c(I+2)} \dots \xrightarrow{r_c(J-1)} p_{J-1}$$
of A_r on r_c, defined as $r_c(u) = (f_1 \star f_2)(u).T^{con}(c)$ for $u \in [0, J-1]$, such that a final state is in $\{p_I, \dots, p_{J-1}\}$ and $p_{J-1} \xrightarrow{r_c(I)} p_I$.

Proof. We have already shown the $\Rightarrow$ direction. For the $\Leftarrow$ direction, define $f = f_1 \star f_2^\omega$, i.e.

$$f(u) = \begin{cases} f_1(u), & \text{if } u < I; \\ f_2((u-I) \bmod (J-I)), & \text{if } u \geq I. \end{cases}$$

It is clear that f is a state function for ϕ. We will show how to find potentially well-founded runs for all the types at all the moments and for all the constants.

- Fix a moment $u \in \mathbb{N}$ and a type $t \in f(u).T$. If $u < I$, then $f(u) = f_1(u) = (f_1 \star f_2)(u)$. If $u \geq I$, then $f(u) = f_2((u-I) \bmod (J-I)) = (f_1 \star f_2)(I + (u-I) \bmod (J-I))$. There is a sequence r' of length J in $\prod_{v\in[0,J)}(f_1 \star f_2)(v).T$ that has the properties stated in (2). Let r_1

be the restriction of r' to $[0, I)$ and r_2 the restriction of r' to $[I, J)$. Define $r = r_1 \star r_2^\omega$, i.e. r is the sequence

$$r(0), r(1), \ldots, r(I-1), \underbrace{r(I), r(I+1), \ldots, r(J-1)}_{\text{period}}, \underbrace{r(I), r(I+1), \ldots, r(J-1)}_{\text{period}}, \ldots$$

The following is a computation of A_r on r.

$$q_r^0 \xrightarrow{r(0)} p_0 \xrightarrow{r(1)} \ldots \xrightarrow{r(I-1)} p_{I-1} \xrightarrow{r(I)} \underbrace{p_I \xrightarrow{r(I+1)} \ldots \xrightarrow{r(J-1)} p_{J-1}}_{\text{final state}} \xrightarrow{r(I)}$$

$$\underbrace{p_I \xrightarrow{r(I+1)} \ldots \xrightarrow{r(J-1)} p_{J-1}}_{\text{final state}} \xrightarrow{r(I)} \underbrace{p_I \xrightarrow{r(I+1)} \ldots \xrightarrow{r(J-1)} p_{J-1}}_{\text{final state}} \xrightarrow{r(I)} \ldots$$

Obviously, the computation is accepting and hence r is a potentially well-founded run in f.

- Fix a constant $c \in \mathsf{con}[\phi]$. Let r_c^1 be the restriction of r_c to $[0, I)$ and r_c^2 be the restriction of r_c to $[I, J)$. As before, we show that $r_c = r_c^1 \star (r_c^2)^\omega$ is a potentially well-founded run in f.

Define $\mathscr{R}$ so as to contain all the above potentially well-founded runs. Observe that

$$\mathscr{R} \subseteq \{r_1 \star r_2^\omega \mid r_1 \text{ is in } \prod_{v \in [0,I)} f(v).T \text{ and } r_2 \text{ is in } \prod_{v \in [I,J)} f(v).T\}$$

and, since $\prod_{v \in [0,I)} f(v).T$ and $\prod_{v \in [I,J-1)} f(v).T$ are finite, $\mathscr{R}$ is finite as well. □

We will introduce a slightly more complex decoration from the one used in Section 7.1. Consider a sequence of state candidates f. We will "mark" the moment w, where ϕ is satisfied, the moment I, where the periodic part starts, and the moment $(J-1)$, where the periodic part ends. We do not need several marks to indicate these moments. Just one will suffice, since we can assume without loss of generality that $w < I < J-1$. For the moments in $[0, I]$, we need the decoration used in Section 7.1. That is, for each moment $u \in [0, I]$ and each type $t \in f(u).T$ we have a decoration $f(u).\pi(t)$, which a subset of Q_r. A state q is in $f(u).\pi(t)$ iff there is a sequence r in $\prod_{v \in [0,u]} f(v).T$ with $r(u) = t$ and there is an A_r-computation

$$q_r^0 \xrightarrow{r(0)} p_0 \xrightarrow{r(1)} p_1 \xrightarrow{r(2)} \ldots \xrightarrow{r(u-1)} p_{u-1} \xrightarrow{r(u)=t} p_u = q$$

on r that ends at state q. For the moments in $[I+1, J-1]$, we need a more detailed decoration. For each moment $u \in [I+1, J-1]$ and each type $t \in f(u).T$ we have a decoration $f(u).\lambda(t)$, which is a subset of $\mathsf{Types}(\phi) \times Q_r \times Q_r \times \{1, 0\}$ (observe that the size of this set is singly exponetial in the size of ϕ). A quadruple (t_I, q_I, q, x) is in $f(u).\lambda(t)$ iff $q_I \in f(I).\pi(t_I)$ and there is a sequence r in $\prod_{v \in [I+1,u]} f(v).T$ with $r(u) = t$ and an A_r-computation

$$\ldots \xrightarrow{t_I} q_I \xrightarrow{r(I+1)} p_{I+1} \xrightarrow{r(I+2)} p_{I+2} \xrightarrow{r(I+3)} \ldots \xrightarrow{r(u-1)} p_{u-1} \xrightarrow{r(u)=t} p_u = q$$

such that $x = 1 \iff$ a final state is in $\{q_I, \ldots, q_u\}$.

Definition 65. An A_r-*decorated state candidate* for ϕ is now defined as a quintuple $(T, T^{con}, m, \pi, \lambda)$, where (T, T^{con}) is a state candidate for ϕ, $m \in \{\star, \llcorner\!\lrcorner\}$ ($\star$ indicates that we have a mark, and $\llcorner\!\lrcorner$ that we have no mark), $\pi : T \to \wp(Q_r)$, and $\lambda : T \to \wp(\mathsf{Types}(\phi) \times Q_r \times Q_r \times \{1, 0\})$. Let $\mathsf{DStateC}(\phi)$ be the set of all A_r-decorated state candidates. Observe that there are at most

$$\natural(\phi) = 2^{\flat(\phi)} \times \underbrace{\flat(\phi) \times \cdots \times \flat(\phi)}_{|\mathsf{con}[\phi]| \text{ times}} \times 2 \times \underbrace{2^{|Q_r|} \times \cdots \times 2^{|Q_r|}}_{\flat(\phi) \text{ times}} \times \underbrace{2^{\flat(\phi) \times |Q_r|^2 \times 2} \times \cdots \times 2^{\flat(\phi) \times |Q_r|^2 \times 2}}_{\flat(\phi) \text{ times}}$$

$$= 2^{\flat(\phi)} \times \flat(\phi)^{|\mathsf{con}[\phi]|} \times 2 \times 2^{|Q_r| \times \flat(\phi)} \times 2^{\flat(\phi) \times |Q_r|^2 \times 2 \times \flat(\phi)}$$

of them. It is easy to see that $|\mathsf{DStateC}(\phi)|$ is doubly exponential in the size of ϕ.

Let f be an infinite sequence in $\mathsf{DStateC}(\phi)^\omega$. We say that f is *correctly decorated* if the following conditions hold.

- There are at least three $\star$ marks. Let $w, I, J-1$ be the first three moments with $w < I < J-1$, where marks appear.

- We are interested in the correctness of the π decoration only at the interval $[0, I]$. So, we require:

$$f(0).\pi(t) = \delta_r(q_r^0, t) \qquad\qquad f(u+1).\pi(t) = \bigcup_{t' \in f(u).T} \Big\{ \bigcup_{q \in f(u).\pi(t')} \delta_r(q, t) \Big\}$$

for $u \in [0, I)$. For the λ decoration we require:

$$f(I+1).\lambda(t) = \bigcup_{\substack{t_I \in \\ f(I).T}} \bigcup_{\substack{q_I \in \\ f(I).\pi(t_I)}} \big\{ (t_I, q_I, q, x) \mid p \in \delta_r(q_I, t), x = 1 \iff (q_I \in F_r \vee q \in F_r) \big\}$$

$$f(u+1).\lambda(t) = \bigcup_{\substack{t' \in \\ f(u).T}} \bigcup_{\substack{(t_I, q_I, q, x) \in \\ f(u).\lambda(t')}} \big\{ (t_I, q_I, p, y) \mid p \in \delta_r(q, t), y = 1 \iff (x = 1 \vee p \in F_r) \big\}$$

for $u \in [I+1, J-1)$.

We define a Büchi automaton A_1 that accepts the ω-language

$$L(A_1) = \{ f \in \mathsf{DStateC}(\phi)^\omega \mid f \text{ is a correctly } A_r\text{-decorated state function that satisfies } \phi \} .$$

Consider the symbols $\{\varphi, \mathsf{I}, \mathsf{J\text{-}1}\}$. They mean that the automaton expects to see a mark for the satisfaction of ϕ, the beginning of the periodic part, and the end of the periodic part respectively. The symbols $\{q_\pi^0, q_\pi^{>0}, q_\lambda^{i+1}, q_\lambda^{>i+1}\}$ should be read as 'checking π decoration at 0', 'checking π decoration in $[1, I]$', 'checking λ decoration at $(I+1)$', and 'checking λ decoration in $[I+2, J-1]$' respectively. For all the decoration checks, apart from the one at the very beginning of the sequence, A_1 needs to remember the entire previous A_r-decorated state candidate. The set of states of A_1 is defined as

$$Q_1 = \big\{ (q_\pi^0, \varphi), q_F, q_A \big\} \cup \big[\big\{ (q_\pi^{>0}, \varphi), (q_\pi^{>0}, \mathsf{I}), (q_\lambda^{i+1}, \mathsf{J\text{-}1}), (q_\lambda^{>i+1}, \mathsf{J\text{-}1}) \big\} \times \mathsf{DStateC}(\phi) \big] .$$

(q_π^0, φ) is the initial state of A_1. It is clear that $|Q_1|$ is doubly exponential in the size of ϕ. The transition function for A_1 is defined as follows.

$$\delta_1(\langle q_\pi^0, \varphi \rangle, \mathfrak{C}) = \begin{cases} (q_\pi^{>0}, \varphi, \mathfrak{C}), & \text{if } \mathfrak{C}.m = \text{␣}, \mathfrak{C} \text{ is realizable and } \mathfrak{C}.\pi \text{ is correct} \\ (q_\pi^{>0}, \mathsf{I}, \mathfrak{C}), & \text{if } \mathfrak{C}.m = \star, \mathfrak{C} \text{ is realizable, } \mathfrak{C}.\pi \text{ is correct, and } \mathfrak{C} \text{ satisfies } \phi \\ q_F, & \text{otherwise} \end{cases}$$

$$\delta_1(q_F, \mathfrak{C}) = \{q_F\}$$

$$\delta_1(\langle q_\pi^{>0}, \varphi, \mathfrak{C} \rangle, \mathfrak{C}') = \begin{cases} (q_\pi^{>0}, \varphi, \mathfrak{C}'), & \text{if } \mathfrak{C}'.m = \text{␣}, \mathfrak{C}' \text{ is realizable and } \mathfrak{C}'.\pi \text{ is correct} \\ (q_\pi^{>0}, \mathsf{I}, \mathfrak{C}'), & \text{if } \mathfrak{C}'.m = \star, \mathfrak{C}' \text{ is realizable, } \mathfrak{C}'.\pi \text{ is correct, and } \mathfrak{C}' \text{ satisfies } \phi \\ q_F, & \text{otherwise} \end{cases}$$

$$\delta_1(\langle q_\pi^{>0}, \mathsf{I}, \mathfrak{C} \rangle, \mathfrak{C}') = \begin{cases} (q_\pi^{>0}, \mathsf{I}, \mathfrak{C}'), & \text{if } \mathfrak{C}'.m = \text{␣}, \mathfrak{C}' \text{ is realizable and } \mathfrak{C}'.\pi \text{ is correct} \\ (q_\lambda^{i+1}, \mathsf{J\text{-}1}, \mathfrak{C}'), & \text{if } \mathfrak{C}'.m = \star, \mathfrak{C}' \text{ is realizable and } \mathfrak{C}'.\pi \text{ is correct} \\ q_F, & \text{otherwise} \end{cases}$$

$$\delta_1(\langle q_\lambda^{i+1}, \mathsf{J\text{-}1}, \mathfrak{C} \rangle, \mathfrak{C}') = \begin{cases} (q_\lambda^{>i+1}, \mathsf{J\text{-}1}, \mathfrak{C}'), & \text{if } \mathfrak{C}'.m = \text{␣}, \mathfrak{C}' \text{ is realizable and } \mathfrak{C}'.\lambda \text{ is correct} \\ q_A, & \text{if } \mathfrak{C}'.m = \star, \mathfrak{C}' \text{ is realizable and } \mathfrak{C}'.\lambda \text{ is correct} \\ q_F, & \text{otherwise} \end{cases}$$

$$\delta_1(\langle q_\lambda^{>i+1}, \mathsf{J\text{-}1}, \mathfrak{C} \rangle, \mathfrak{C}') = \begin{cases} (q_\lambda^{>i+1}, \mathsf{J\text{-}1}, \mathfrak{C}'), & \text{if } \mathfrak{C}'.m = \text{␣}, \mathfrak{C}' \text{ is realizable and } \mathfrak{C}'.\lambda \text{ is correct} \\ q_A, & \text{if } \mathfrak{C}'.m = \star, \mathfrak{C}' \text{ is realizable and } \mathfrak{C}'.\lambda \text{ is correct} \\ q_F, & \text{otherwise} \end{cases}$$

$$\delta_1(q_A) = q_A$$

The set of final states is $F_1 = \{q_A\}$.

We define the alternating Büchi automaton $A_2 = (Q_2, \mathsf{DStateC}(\phi), (q_0, \varphi), \delta_2, F_2)$, which accepts the ω-language $L(A_2)$, for which it holds that for any $f \in L(A_1)$,

$$f \in L(A_2) \iff f \text{ can be extended to a well-founded finitary quasimodel .}$$

The way the automaton works can be thought of as the execution of a "main" process that launches a new process for each check that has to be performed to satisfy the conditions of Proposition 64. That is, for each moment u and each type $t \in f(u).T$, a process is launched that checks for the existence of an appropriate sequence in $\prod_{v\in[0,J)} f(v).T$ that goes through t at u. Moreover, a process is launched for each constant c that checks whether the sequence r_c is appropriate. The set of states of A_2 is defined as

$$\begin{aligned} Q_2 = &\big\{(q_0,\varphi),(q_1,\varphi),(q_1,\mathsf{I}),(q_1,\mathsf{J\text{-}1}),q_F,q_A\big\}\cup \\ &\big(Q_r \times \{\varphi,\mathsf{I}\}\big) \cup \big(\mathsf{Types}(\phi) \times Q_r \times Q_r \times \{1,0\}\big)\cup \\ &\big(\{\mathsf{Check}\} \times \mathsf{Types}(\phi) \times Q_r \times Q_r \times \{1,0\}\big)\cup \\ &\big(\mathsf{con}[\phi] \times Q_r \times \{\varphi,\mathsf{I}\}\big) \cup \big(\mathsf{con}[\phi] \times \mathsf{Types}(\phi) \times Q_r \times Q_r \times \{1,0\}\big)\ . \end{aligned}$$

The main process involves the states (q_0,φ), (q_1,φ), (q_1,I), and $(q_1,\mathsf{J\text{-}1})$. The transition function for these states is defined as

$$\delta_2(\langle q_0,\varphi\rangle,\mathfrak{C}) = \begin{cases} \bigwedge\limits_{c\in\mathsf{con}[\phi]} \Bigg\{ \bigvee\limits_{\substack{q\in\mathfrak{C}.\pi(t_c)\\ t_c=\mathfrak{C}.T^{con}(c)}} (c,q,\varphi) \Bigg\} \wedge \bigwedge\limits_{t\in\mathfrak{C}.T} \Bigg\{ \bigvee\limits_{q\in\mathfrak{C}.\pi(t)} (q,\varphi) \Bigg\} \wedge (q_1,\varphi), & \text{if } \mathfrak{C}.m = \llcorner\!\lrcorner \\ \bigwedge\limits_{c\in\mathsf{con}[\phi]} \Bigg\{ \bigvee\limits_{\substack{q\in\mathfrak{C}.\pi(t_c)\\ t_c=\mathfrak{C}.T^{con}(c)}} (c,q,\mathsf{I}) \Bigg\} \wedge \bigwedge\limits_{t\in\mathfrak{C}.T} \Bigg\{ \bigvee\limits_{q\in\mathfrak{C}.\pi(t)} (q,\mathsf{I}) \Bigg\} \wedge (q_1,\mathsf{I}), & \text{if } \mathfrak{C}.m = \star \end{cases}$$

$$\delta_2(\langle q_1,\varphi\rangle,\mathfrak{C}) = \begin{cases} \bigwedge\limits_{t\in\mathfrak{C}.T} \Bigg\{ \bigvee\limits_{q\in\mathfrak{C}.\pi(t)} (q,\varphi) \Bigg\} \wedge (q_1,\varphi), & \text{if } \mathfrak{C}.m = \llcorner\!\lrcorner \\ \bigwedge\limits_{t\in\mathfrak{C}.T} \Bigg\{ \bigvee\limits_{q\in\mathfrak{C}.\pi(t)} (q,\mathsf{I}) \Bigg\} \wedge (q_1,\mathsf{I}), & \text{if } \mathfrak{C}.m = \star \end{cases}$$

$$\delta_2(\langle q_1,\mathsf{I}\rangle,\mathfrak{C}) = \begin{cases} \bigwedge\limits_{t\in\mathfrak{C}.T} \Bigg\{ \bigvee\limits_{q\in\mathfrak{C}.\pi(t)} (q,\mathsf{I}) \Bigg\} \wedge (q_1,\mathsf{I}), & \text{if } \mathfrak{C}.m = \llcorner\!\lrcorner \\ \bigwedge\limits_{t_I\in\mathfrak{C}.T} \Bigg\{ \bigvee\limits_{q_I\in\mathfrak{C}.\pi(t_I)} \langle t_I,q_I,q_I,\mathsf{isFinal}(q_I)\rangle \Bigg\} \wedge (q_1,\mathsf{J\text{-}1}), & \text{if } \mathfrak{C}.m = \star \end{cases}$$

$$\delta_2(\langle q_1,\mathsf{J\text{-}1}\rangle,\mathfrak{C}) = \begin{cases} \bigwedge\limits_{t\in\mathfrak{C}.T} \Bigg\{ \bigvee\limits_{\substack{(t_I,q_I,q,x)\in\\ \mathfrak{C}.\lambda(t)}} (t_I,q_I,q,x) \Bigg\} \wedge (q_1,\mathsf{J\text{-}1}), & \text{if } \mathfrak{C}.m = \llcorner\!\lrcorner \\ \bigwedge\limits_{t\in\mathfrak{C}.T} \Bigg\{ \bigvee\limits_{\substack{(t_I,q_I,q,x)\in\\ \mathfrak{C}.\lambda(t)}} (\mathsf{Check},t_I,q_I,q,x) \Bigg\}, & \text{if } \mathfrak{C}.m = \star \end{cases}$$

We define $\mathsf{isFinal}(q) = 1$, whenever $q \in F_r$, and $\mathsf{isFinal}(q) = 0$, whenever $q \notin F_r$. The processes for the constants are launched from the initial state (q_0,φ). The processes for the types are launched from (q_0,φ), (q_1,φ), (q_1,I), $(q_1,\mathsf{J\text{-}1})$. Notice that depending on whether we are in the periodic part or not, these processes are initialized differently.

A process for a type that is not in the periodic part yet needs to remember only the current state of the A_r computation it is simulating. When it enters the periodic part it also needs to remember the type (t_I) and state (q_I) it went through at moment I and whether it has seen an A_r-accepting state in the interval $[I, \text{current-moment}]$ ($x = 1$ if yes, $x = 0$ if no). This extra information will be used when the mark for the end of the periodic mark is seen in order to decide whether the process will accept or not. When presented with the next A_r-decorated state candidate, the process guesses the type it should continue on and also guesses the next A_r-state from the transition function δ_r.

$$\delta_2(\langle q,\varphi\rangle,\mathfrak{C}) = \begin{cases} \bigvee_{t\in\mathfrak{C}.T}\ \bigvee_{q'\in\delta_r(q,t)} (q',\varphi), & \text{if } \mathfrak{C}.m = \llcorner\!\lrcorner \\ \bigvee_{t\in\mathfrak{C}.T}\ \bigvee_{q'\in\delta_r(q,t)} (q',\mathsf{I}), & \text{if } \mathfrak{C}.m = \star \end{cases}$$

$$\delta_2(\langle q,\mathsf{I}\rangle,\mathfrak{C}) = \begin{cases} \bigvee_{t\in\mathfrak{C}.T}\ \bigvee_{q'\in\delta_r(q,t)} (q',\mathsf{I}), & \text{if } \mathfrak{C}.m = \llcorner\!\lrcorner \\ \bigvee_{t_I\in\mathfrak{C}.T}\ \bigvee_{q_I\in\delta_r(q,t_I)} \langle t_I,q_I,q_I,\mathsf{isFinal}(q_I)\rangle, & \text{if } \mathfrak{C}.m = \star \end{cases}$$

$$\delta_2(\langle t_I,q_I,q,x\rangle,\mathfrak{C}) = \begin{cases} \bigvee_{t\in\mathfrak{C}.T}\ \bigvee_{q'\in\delta_r(q,t)} \langle t_I,q_I,q',\mathsf{hasFinal}(x,q')\rangle, & \text{if } \mathfrak{C}.m = \llcorner\!\lrcorner \\ \bigvee_{t\in\mathfrak{C}.T}\ \bigvee_{q'\in\delta_r(q,t)} \langle \mathsf{Check},t_I,q_I,q',\mathsf{hasFinal}(x,q')\rangle, & \text{if } \mathfrak{C}.m = \star \end{cases}$$

We define $\mathsf{hasFinal}(x,q) = 1$, whenever $x = 1$ or $q \in F_r$, and $\mathsf{hasFinal}(x,q) = 0$, whenever $x = 0$ and $q \notin F_r$. When the end of the periodic part has been reached, we check whether the sequence that the process has been following fulfills the required conditions. Observe that from this point on the state candidates that come up are not taken into account for the transition table of the automaton. The decision of whether the automaton accepts depends entirely on the marked finite prefix.

$$\delta_2(\langle \mathsf{Check},t_I,q_I,q,x\rangle,\mathfrak{C}) = \begin{cases} q_A, & \text{if } x=1 \text{ and } q_I\in\delta_r(q,t_I) \\ q_F, & \text{if } x=0 \text{ or } q_I\notin\delta_r(q,t_I) \end{cases}$$

$$\delta_2(q_A,\mathfrak{C}) = q_A$$

$$\delta_2(q_F,\mathfrak{C}) = q_F$$

A process for a constant c does not have to guess the next type. It reads the next type from $\mathfrak{C}.T^{con}(c)$. So, the transition function is defined as follows.

$$\delta_2(\langle c,q,\varphi\rangle,\mathfrak{C}) = \begin{cases} \bigvee_{q'\in\delta_r(q,\mathfrak{C}.T^{con}(c))} (c,q',\varphi), & \text{if } \mathfrak{C}.m = \llcorner\!\lrcorner \\ \bigvee_{q'\in\delta_r(q,\mathfrak{C}.T^{con}(c))} (c,q',\mathsf{I}), & \text{if } \mathfrak{C}.m = \star \end{cases}$$

$$\delta_2(\langle c,q,\mathsf{I}\rangle,\mathfrak{C}) = \begin{cases} \bigvee_{q'\in\delta_r(q,\mathfrak{C}.T^{con}(c))} (c,q',\mathsf{I}), & \text{if } \mathfrak{C}.m = \llcorner\!\lrcorner \\ \bigvee_{q_I\in\delta_r(q,\mathfrak{C}.T^{con}(c))} (c,\mathfrak{C}.T^{con}(c),q_I,q_I,\mathsf{isFinal}(q_I)), & \text{if } \mathfrak{C}.m = \star \end{cases}$$

$$\delta_2(\langle c,t_I,q_I,q,x\rangle,\mathfrak{C}) = \begin{cases} \bigvee_{q'\in\delta_r(q,\mathfrak{C}.T^{con}(c))} \langle c,t_I,q_I,q',\mathsf{hasFinal}(x,q')\rangle, & \text{if } \mathfrak{C}.m = \llcorner\!\lrcorner \\ \bigvee_{q'\in\delta_r(q,\mathfrak{C}.T^{con}(c))} \langle \mathsf{Check},t_I,q_I,q',\mathsf{hasFinal}(x,q')\rangle, & \text{if } \mathfrak{C}.m = \star \end{cases}$$

The set of final states is $F_2 = \{q_A\}$.

It is easy to see that $|Q_2|$ is exponential in the size of ϕ. We then construct a nondeterministic automaton A_2', which is language-equivalent to A_2. This involves an exponential blow-up in the number of states. So, the number of states of A_2' is doubly exponential in the size of ϕ. With a quadratic construction, we get the automaton A' that accepts the language

$$L(A') = L(A_1)\cap L(A_2') = L(A_1)\cap L(A_2) = \{f \in \mathsf{DStateC}(\phi)^\omega \mid \\ f \text{ can be extended to a well-founded finitary quasimodel that satisfies } \phi\}\ .$$

The projection of $L(A')$ on $\mathsf{StateC}(\phi)$ is accepted by the automaton A, which guesses the decorations. The number of states of A is doubly exponential in the size of ϕ.

Theorem 66. Let $\mathcal{F}$ be a sublanguage of $\mathcal{FOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}$.

- Suppose that there is an algorithm which, given a state candidate $\mathfrak{C}$ for a $\mathcal{F}$-sentence, can decide whether $\mathfrak{C}$ is finitely realizable using exponential space in the size of ϕ. Then, the finite satisfiability problem for $\mathcal{F}$-sentences is in $\mathsf{EXPSPACE}$.
- Suppose that there is an algorithm which, given a state candidate $\mathfrak{C}$ for a $\mathcal{F}$-sentence, can decide whether $\mathfrak{C}$ is finitely realizable using doubly exponential time in the size of ϕ. Then, the finite satisfiability problem for $\mathcal{F}$-sentences is in $\mathsf{2EXPTIME}$.

Proof. Similar to the proof of Theorem 63. $\square$

8. Monodic packed fragment

Most of the decidability results for monodic fragments concern logics without equality. It is known that even the simple monodic two-variable fragment with equality over the naturals is not even recursively enumerable [12]. Hodkinson considers in [28] the monodic packed fragment with Until and Since, which is a generalization of the packed fragment of first-order logic, introduced in [41], and shows that it is decidable over the naturals, the integers, the rationals, the class of all finite strict linear orders or the class of any first-order definable class of strict linear orders. Following closely the presentation in [28], we discuss here the modifications of the quasimodel technique required to extend our complexity results to the monodic packed case over the naturals.

Remember that a crucial point in the proof of "well-founded quasimodel implies model" (Theorem 56) is blowing-up the first-order structure $\mathfrak{D}_u$ that realizes the quasistate $f(u)$ to a first-order structure $\mathfrak{E}_u$ that also realizes the quasistate $f(u)$. This is necessary in order to to get a family of structures $\{\mathfrak{E}_u\}_{\mathbb{N}}$ that share the same domain and interpret constants rigidly and thus construct the temporal model. The argument involves the use of Lemma 54, which fails when we add equality to the language. We will proceed to see what further restrictions we need to consider, in order to be able to prove a similar lemma when we include equality.

Let $\mathfrak{D} = (\mathcal{D}, \cdot^{\mathfrak{D}})$, $\mathfrak{E} = (\mathcal{E}, \cdot^{\mathfrak{E}})$ be σ-structures and $R \subseteq \mathcal{D} \times \mathcal{E}$ be an left-total, injective, and right-total relation. As discussed in Remark 43, if we have aRb_1, aRb_2, and $b_1 \neq b_2$, we get that $h = \{(x,a),(y,a)\}R\{(x,b_1),(y,b_2)\} = g$, $\mathfrak{D}, h \models x = y$, and $\mathfrak{E}, g \models x \neq y$. So, we should not allow the variable assignment g to have values that are in the same $\sim_R$-class. We say that a set $B \subseteq \mathcal{E}$ is *R-thin* if for every $b_1, b_2 \in B$ with $b_1 \neq b_2$, $b_1 \not\sim_R b_2$. A variable assignment $g : \mathcal{V} \to \mathcal{E}$ is *R-thin* if $\mathsf{Range}(g)$ (the range of g) is R-thin. Let c be a constant in the signature and assume that $c^{\mathfrak{D}} = a$, aRb_1, aRb_2, $c^{\mathfrak{E}} = b_1$, and $b_1 \neq b_2$. Immediately, we see that $\mathfrak{D}, x \mapsto a \models x = c$ and $\mathfrak{E}, x \mapsto b_2 \not\models x = c$. It follows that we should also require that domain elements in $\mathcal{D}$ that are interpretations of constants are not related to more than one element in $\mathcal{E}$. A similar restriction is required for function symbols as well. Assuming that a_1Rb_1, a_2Rb_2, $a = f^{\mathfrak{D}}(a_1,a_2)Rf^{\mathfrak{E}}(b_1,b_2) = e_1$, aRe_2, and $e_1 \neq e_2$, we have that

$$\mathfrak{D}, \{(x,a_1),(y,a_2),(z,a)\} \models z = f(x,y)$$
$$\mathfrak{E}, \{(x,b_1),(y,b_2),(z,e_2)\} \not\models z = f(x,y)\ .$$

So, every element in the range of the interpretation of some function symbol under $\mathfrak{D}$ should be related to exactly one element of $\mathcal{E}$. With these restrictions, we are able to show the following.

Claim 67. Let $\mathfrak{D}$, $\mathfrak{E}$ be first-order structures over σ and $R \subseteq \mathcal{D} \times \mathcal{E}$ be a left-total, injective, and right-total relation. Assume that the conditions (i), (ii), (iii), (iv) of Lemma 54 hold as well as the following.

(v) For any constant c, and any $b_1, b_2 \in \mathcal{E}$, $c^{\mathfrak{D}}Rb_1$ and $c^{\mathfrak{D}}Rb_2$ implies that $b_1 = b_2$.
(vi) For any function symbol f, any $a \in \mathsf{Range}(f)$, and any $b_1, b_2 \in \mathcal{E}$, aRb_1 and aRb_2 implies that $b_1 = b_2$.

Define $\mathcal{F}$ to be the set of first-order formulas that contains the atomic formulas and is closed under $\neg, \wedge$ (that is, we have no quantification). The following hold.

(1) For any variable assignments $h : \mathcal{V} \to \mathcal{D}$, $g : \mathcal{V} \to \mathcal{E}$, with hRg and g R-thin, and any terms t_1, t_2, $[\![t_1]\!]^{\mathfrak{D}}_h = [\![t_2]\!]^{\mathfrak{D}}_h \iff [\![t_1]\!]^{\mathfrak{E}}_g = [\![t_2]\!]^{\mathfrak{E}}_g$.
(2) For any $\phi \in \mathcal{F}$, and any variable assignments $h : \mathcal{V} \to \mathcal{D}$, $g : \mathcal{V} \to \mathcal{E}$, with hRg and g R-thin, $\mathfrak{D}, h \models \phi \iff \mathfrak{E}, g \models \phi$.

Proof. We show the first claim. Fix h, g, t_1, t_2. From Lemma 54, we have that $[\![t_1]\!]^{\mathfrak{D}}_h R [\![t_1]\!]^{\mathfrak{E}}_g$ and $[\![t_2]\!]^{\mathfrak{D}}_h R [\![t_2]\!]^{\mathfrak{E}}_g$. The $\Leftarrow$ direction follows easily: Since R is injective, $[\![t_1]\!]^{\mathfrak{E}}_g = [\![t_2]\!]^{\mathfrak{E}}_g$ implies that $[\![t_1]\!]^{\mathfrak{D}}_h = [\![t_2]\!]^{\mathfrak{D}}_h$. For the $\Rightarrow$ direction, we will have to examine all possible cases.

- $x = y$. Suppose that $h(x) = h(y)$. Since $h(x)Rg(x)$ and $h(y)Rg(y)$, we have that $g(x) \sim_R g(y)$. From R-thinness of g, we get that $g(x) = g(y)$.
- $x = c$. Suppose that $h(x) = c^{\mathfrak{D}}$. From $c^{\mathfrak{D}} = h(x)Rg(x)$ and $c^{\mathfrak{D}}Rc^{\mathfrak{E}}$, we deduce that $g(x) = c^{\mathfrak{E}}$.

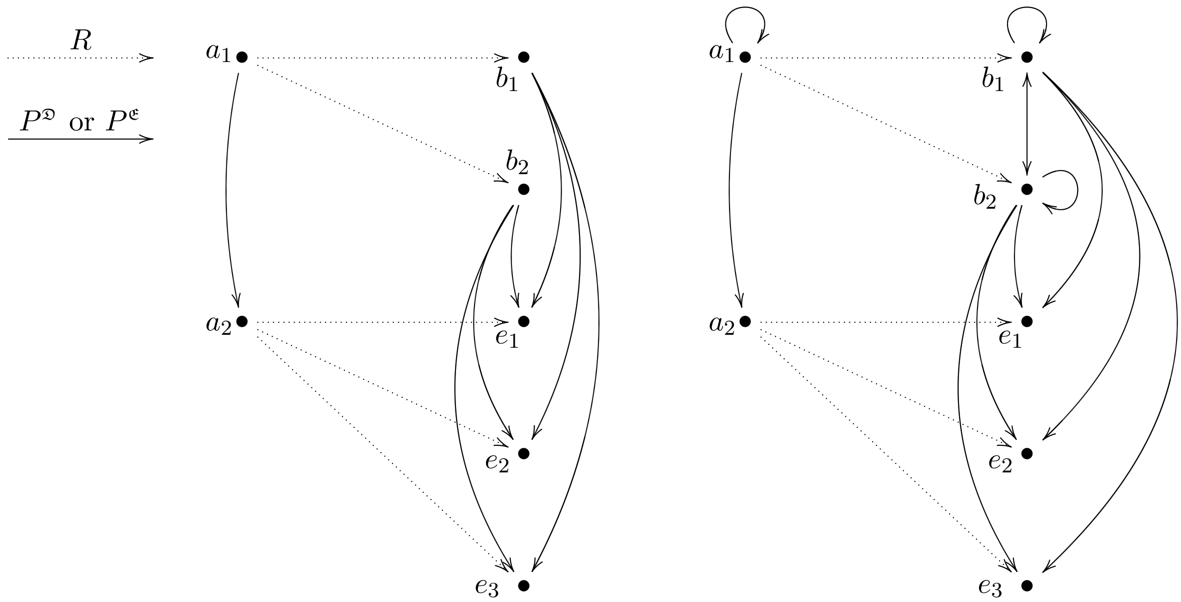

FIGURE 8. Pairs of first-order structures that satisfy the conditions of Lemma 54.

– $x = f(t_1, \ldots, t_m)$. Suppose that $h(x) = f^{\mathfrak{D}}([\![t_1]\!]^{\mathfrak{D}}_h, \ldots, [\![t_m]\!]^{\mathfrak{D}}_h)$. We have that $h(x) \in \mathsf{Range}(f^{\mathfrak{D}})$, $h(x) R g(x)$ and $f^{\mathfrak{D}}([\![t_1]\!]^{\mathfrak{D}}_h, \ldots, [\![t_m]\!]^{\mathfrak{D}}_h) R f^{\mathfrak{E}}([\![t_1]\!]^{\mathfrak{E}}_g, \ldots, [\![t_m]\!]^{\mathfrak{E}}_g)$. It follows that $g(x) = f^{\mathfrak{E}}([\![t_1]\!]^{\mathfrak{E}}_g, \ldots, [\![t_m]\!]^{\mathfrak{E}}_g)$.
– The cases $c_1 = c_2$, $c = f(t_1, \ldots, t_m)$, $f_1(t_1, \ldots, t_m) = f_2(\tau_1, \ldots, \tau_k)$ involve similar arguments.

We proceed by induction on ϕ for the second claim.

- The atomic case $t_1 = t_2$ is an easy consequence of the first claim. Fix h, g. We have that $\mathfrak{D}, h \models t_1 = t_2 \iff [\![t_1]\!]^{\mathfrak{D}}_h = [\![t_2]\!]^{\mathfrak{D}}_h \iff [\![t_1]\!]^{\mathfrak{E}}_g = [\![t_2]\!]^{\mathfrak{E}}_g \iff \mathfrak{E}, g \models t_1 = t_2$.
- The case $\phi = p$ is trivial.
- $\phi = P(t_1, \ldots, t_n)$. Fix h, g. Easily, we get that $\mathfrak{D}, h \models P(t_1, \ldots, t_n) \iff ([\![t_1]\!]^{\mathfrak{D}}_h, \ldots, [\![t_n]\!]^{\mathfrak{D}}_h) \in P^{\mathfrak{D}} \iff ([\![t_1]\!]^{\mathfrak{E}}_g, \ldots, [\![t_n]\!]^{\mathfrak{E}}_g) \in P^{\mathfrak{E}} \iff \mathfrak{E}, g \models P(t_1, \ldots, t_n)$.
- The cases $\phi = \neg\phi_1, (\phi_1 \wedge \phi_2)$ are easy. □

The difficult part is to deal with quantification. In Figure 8, we see two examples of pairs of first-order structures (over a signature that has only one binary predicate symbol P) with domains $\mathcal{D} = \{a_1, a_2\}$, $\mathcal{E} = \{b_1, b_2, e_1, e_2, e_3\}$. For the first example, we have that

$$\mathfrak{D} \not\models \exists x \exists y \exists z (Pxz \wedge Pyz \wedge x \neq y \wedge y \neq z \wedge x \neq z)$$
$$\mathfrak{E} \models \exists x \exists y \exists z (Pxz \wedge Pyz \wedge x \neq y \wedge y \neq z \wedge x \neq z) \ .$$

Observe that a variable assignment that makes the formula

$$Pxz \wedge Pyz \wedge x \neq y \wedge y \neq z \wedge x \neq z$$

true under $\mathfrak{E}$ is $\{(x, b_1), (y, b_2), (z, e_1)\}$, which is clearly not R-thin. For the second example, we have that

$$\mathfrak{D} \not\models \exists x \exists y (Pxx \wedge Pyy \wedge Pxy \wedge x \neq y)$$
$$\mathfrak{E} \models \exists x \exists y (Pxx \wedge Pyy \wedge Pxy \wedge x \neq y) \ .$$

Again, we notice that the variable assignment $\{(x, b_1), (y, b_2)\}$ which makes the formula

$$Pxx \wedge Pyy \wedge Pxy \wedge x \neq y$$

true under $\mathfrak{E}$ is not R-thin. We want to enforce that any formula of the form $\exists x_1 \ldots \exists x_k \phi$ is true under $\mathfrak{E}, g$ if and only if ϕ is true under $\mathfrak{E}, g' = g[x_1 \mapsto e_1, \ldots, x_k \mapsto e_k]$, where g' is R-thin. This motivates the following definition.

Definition 68 (the packed fragment of first-order logic [41])**.** We say that a formula ϕ *packs* a finite set of variables $\{x_1, \ldots, x_k\}$ if

– $\mathsf{fvars}[\phi] = \{x_1, \dots, x_k\}$,
– ϕ is a conjunction of atomic formulas (including equalities) or existentially quantified atomic formulas, and
– any two distinct variables in $\{x_1, \dots, x_k\}$ occur free in some conjunct of ϕ.

We say that ϕ is a *packing guard* if ϕ packs $\mathsf{fvars}[\phi]$. The *packed fragment* is the smallest set $\mathcal{PFOL}$ of first-order formulas that satisfies the following conditions.

- $\mathcal{PFOL}$ contains all atomic $\mathcal{FOL}$-formulas, including equalities.
- $\mathcal{PFOL}$ is closed under $\neg$ and $\wedge$.
- (*packed quantification*) If ϕ is a $\mathcal{PFOL}$-formula, γ is a packing guard, $\mathsf{fvars}[\phi] \subseteq \mathsf{fvars}[\gamma]$, and $x_1, \dots, x_n$ are individual variables, then $\exists \bar{x}(\gamma \wedge \phi)$ is a $\mathcal{PFOL}$-formula. $\exists \bar{x}$ abbreviates $\exists x_1 \dots \exists x_n$. We say that γ is the guard of $\exists \bar{x}(\gamma \wedge \phi)$.

The idea is that the guards together with some additional restrictions on the interpretations of the predicate symbols under $\mathfrak{E}$ will give us the desired R-thinness of the variable assignments. We require that for any n-ary predicate symbol P, the interpretation $P^{\mathfrak{E}}$ relates elements of $\mathcal{E}$ that form an R-thin set. That is, for any $e_1, \dots, e_n \in \mathcal{E}$, $P^{\mathfrak{E}}(e_1, \dots, e_n)$ implies that $\{e_1, \dots, e_n\}$ is an R-thin set. We strengthen Claim 67 to include all packed formulas.

Lemma 69. Let $\mathfrak{D} = (\mathcal{D}, \cdot^{\mathfrak{D}})$, $\mathfrak{E} = (\mathcal{E}, \cdot^{\mathfrak{E}})$ be first-order σ-structures and $R \subseteq \mathcal{D} \times \mathcal{E}$ be a left-total, injective, and right-total relation. Assume that the conditions (ii), (iii), (iv) of Lemma 54, and the conditions (v), (vi) of Claim 67 hold. We replace condition (i) of Lemma 54 by the following.

(vii) For any predicate symbol $P \in \mathcal{P}$ with $n = \mathsf{ar}[P]$ and any $(b_1, \dots, b_n) \in \mathcal{E}^n$,
$$(b_1, \dots, b_n) \in P^{\mathfrak{E}} \iff \{b_1, \dots, b_n\} \text{ is } R\text{-thin } \& \ (R^{-1}(b_1), \dots, R^{-1}(b_n)) \in P^{\mathfrak{D}} \ .$$

For any $\mathcal{PFOL}[\sigma]$-formula ϕ, and any individual variable assignments $h : \mathsf{fvars}[\phi] \to \mathcal{D}$, $g : \mathsf{fvars}[\phi] \to \mathcal{E}$, if hRg and g is R-thin, then $\mathfrak{D}, h \models \phi \iff \mathfrak{E}, g \models \phi$.

Proof. Fix $\mathfrak{D}, \mathfrak{E}, R$. The proof proceeds by induction on ϕ.

- $\phi = P(t_1, \dots, t_n)$. Fix h, g with hRg and g R-thin. Suppose that $\mathfrak{D}, h \models P(t_1, \dots, t_n)$, which means that $([\![t_1]\!]^{\mathfrak{D}}_h, \dots, [\![t_n]\!]^{\mathfrak{D}}_h) \in P^{\mathfrak{D}}$. We know that $[\![t_1]\!]^{\mathfrak{D}}_h R [\![t_1]\!]^{\mathfrak{E}}_g, \dots, [\![t_n]\!]^{\mathfrak{D}}_h R [\![t_n]\!]^{\mathfrak{E}}_g$. Observe that $\{[\![t_1]\!]^{\mathfrak{E}}_g, \dots, [\![t_n]\!]^{\mathfrak{E}}_g\}$ is R-thin: for any terms t_1, t_2, $[\![t_1]\!]^{\mathfrak{E}}_g \sim_R [\![t_2]\!]^{\mathfrak{E}}_g$ implies that $[\![t_1]\!]^{\mathfrak{E}}_g = [\![t_2]\!]^{\mathfrak{E}}_g$, as we verify by considering all the possible cases. It follows that $([\![t_1]\!]^{\mathfrak{E}}_g, \dots, [\![t_n]\!]^{\mathfrak{E}}_g) \in P^{\mathfrak{E}}$ and hence $\mathfrak{E}, g \models P(t_1, \dots, t_n)$. The converse is very straightforward.
- The atomic case $\phi = p$ is trivial and the case $t_1 = t_2$ is dealt with as in Claim 67.
- The cases $\phi = \neg \phi_1, (\phi_1 \wedge \phi_2)$ are easy.
- $\phi = \exists x_1 \dots \exists x_n(\gamma \wedge \phi_1)$. Fix h, g with hRg and g R-thin. Without loss of generality, we can assume that $\{x_1, \dots, x_n\} \subseteq \mathsf{fvars}[\gamma]$.

  Suppose that $\mathfrak{D}, h \models \phi$. There are $a_1, \dots, a_n \in \mathcal{D}$ such that $\mathfrak{D}, h' = h[x_1 \mapsto a_1, \dots, x_n \mapsto a_n] \models \gamma \wedge \phi_1$. We will define g' so that it agrees with g on $\mathsf{fvars}[\phi]$, $h'Rg'$, and the values for $x_1, \dots, x_n$ are chosen in a way that does not destroy R-thinness. Let $f : \mathsf{Range}(h) \to \mathsf{Range}(g)$ be defined as $f = \{\langle h(x), g(x) \rangle \mid x \in \mathcal{V}\}$. By R-thinness of g, f is a function and $\mathsf{Range}(f)$ is R-thin. Extend f to f' with $\mathsf{Domain}(f') = \mathsf{Range}(h) \cup \{a_1, \dots, a_n\}$ so that $\mathsf{Range}(f')$ is R-thin: take $a \in \{a_1, \dots, a_n\} \setminus \mathsf{Domain}(f)$, pick any b with aRb, and put $f'(a) = b$. Then, we can define g' as $g'(x) = f'(h'(x))$ for all $x \in \mathsf{fvars}[\gamma]$. It is easy to verify that $\mathfrak{E}, g' \models \gamma$. From the inductive hypothesis, we get that $\mathfrak{E}, g' \models \phi_1$. Therefore, $\mathfrak{E}, g' \models \gamma \wedge \phi_1$ and $\mathfrak{E}, g \models \phi$.

  Conversely, assume that $\mathfrak{E}, g \models \phi$, where g is R-thin. There are $e_1, \dots, e_n \in \mathcal{E}$ such that $\mathfrak{E}, g' = g[x_1 \mapsto e_1, \dots, x_n \mapsto e_n] \models \gamma \wedge \phi_1$. We show that g' is R-thin. For contradiction, assume that there are $x, y \in \mathsf{fvars}[\gamma]$ such that $g'(x) \neq g'(y)$ and $g'(x) \sim_R g'(y)$. There is a conjunct of γ in which both x, y appear free. If the conjunct is the equality $x = y$, then $g'(x) = g'(y)$, which is a contradiction. If it is of the form $P(\dots, x, \dots, y, \dots)$, then we get a contradiction by the R-thinness of $\{\dots, g'(x), \dots, g'(y), \dots\}$. Even for quantified conjuncts of the form $\exists \bar{z} P(\dots, x, \dots, y, \dots)$, the same argument applies. Define $h'(x) = R^{-1}(g'(x))$ for all $x \in \mathsf{fvars}[\gamma]$ and observe that h' agrees with h on $\mathsf{fvars}[\phi]$. Easily, we get that $\mathfrak{D}, h' \models \gamma$. The inductives hypothesis gives us that $\mathfrak{D}, h' \models \phi_1$. Immediately, we deduce that $\mathfrak{D}, h \models \phi$. $\square$

Definition 70 (packing $\exists$-guard, packing $\forall$-guard, $\mathcal{PFOTL}_{\mu\nu}$, $\mathcal{PFOTL}^{\mathsf{pnf}}_{\mu\nu}$, $\mathcal{PFOTL}\mathbf{1}_{\mu\nu}$, $\mathcal{PFOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}$)**.** Let $\sigma = (\mathcal{P}, \mathcal{P}_0, \mathcal{F} = \emptyset, \mathcal{C}, \mathsf{ar})$ be a signature with no function symbols. We say that ϕ is a *packing $\exists$-guard* if it is a packing guard. Dually, we define *packing $\forall$-guards.* A formula γ is said to be a packing $\forall$-guard if it is a disjunction of negated atomic formulas (including inequalities) and universally quantified negated atomic formulas such that any two distinct variables in $\mathsf{fvars}[\gamma]$ occur free in some disjunct of ϕ.

The *packed fragment* of first-order temporal logic with fixpoint operators is the smallest set $\mathcal{PFOTL}_{\mu\nu}$ that satisfies the following conditions.

- $\mathcal{PFOTL}_{\mu\nu}$ includes $\mathcal{P}_0$ and the set of fixpoint variables $\mathcal{X}$.
- For any predicate symbol P, if $n = \mathsf{ar}[P]$ and $t_1, \ldots, t_n$ are σ-terms, then $P(t_1, \ldots, t_n) \in \mathcal{PFOTL}_{\mu\nu}$.
- If t_1, t_2 are σ-terms, then $t_1 = t_2$ is in $\mathcal{PFOTL}_{\mu\nu}$.
- For any unary operator $\circ \in \{\neg, \bigcirc, \bullet, \ominus\}$, if $\phi \in \mathcal{PFOTL}_{\mu\nu}$, then $\circ\phi \in \mathcal{PFOTL}_{\mu\nu}$.
- For any binary operator $\otimes \in \{\wedge, \vee\}$, if $\phi_1, \phi_2 \in \mathcal{PFOTL}_{\mu\nu}$, then $(\phi_1 \otimes \phi_2) \in \mathcal{PFOTL}_{\mu\nu}$.
- (*existential packed quantification*) If ϕ is a $\mathcal{PFOTL}_{\mu\nu}$-formula, γ is a packing $\exists$-guard, $\mathsf{fvars}[\phi] \subseteq \mathsf{fvars}[\gamma]$, and $x_1, \ldots, x_n$ are individual variables, then $\exists\bar{x}(\gamma \wedge \phi)$ is a $\mathcal{PFOTL}_{\mu\nu}$-formula.
- (*universal packed quantification*) If ϕ is a $\mathcal{PFOTL}_{\mu\nu}$-formula, γ is a packing $\forall$-guard, $\mathsf{fvars}[\phi] \subseteq \mathsf{fvars}[\gamma]$, and $x_1, \ldots, x_n$ are individual variables, then $\forall\bar{x}(\gamma \vee \phi)$ is a $\mathcal{PFOTL}_{\mu\nu}$-formula.
- For any fixpoint operator $f \in \{\mu, \nu\}$, if X is a fixpoint variable, $\phi \in \mathcal{PFOTL}_{\mu\nu}$, and all free occurrences of X in ϕ are positive, then $fX\phi \in \mathcal{PFOTL}_{\mu\nu}$.

$\mathcal{PFOTL}^{\mathsf{pnf}}_{\mu\nu}$ is the set of $\mathcal{PFOTL}_{\mu\nu}$-formulas in positive normal form. $\mathcal{PFOTL}\mathbf{1}_{\mu\nu}$ is the set of monodic $\mathcal{PFOTL}_{\mu\nu}$-formulas and $\mathcal{PFOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}$ the set of monodic $\mathcal{PFOTL}^{\mathsf{pnf}}_{\mu\nu}$-formulas.

Notice that a $\mathcal{PFOTL}_{\mu\nu}$-formula ϕ with $\mathsf{fvars}[\phi] = \{x_1, \ldots, x_k\}$ is satisfiable if and only if the $\mathcal{PFOTL}_{\mu\nu}$-dom-sentence $\exists x_1 \ldots \exists x_k(Qx_1 \ldots x_k \wedge \phi)$ is satisfiable, where Q is a k-ary predicate symbol that does not appear in ϕ. It is clear that $\mathcal{PFOTL}_{\mu\nu}$, $\mathcal{PFOTL}^{\mathsf{pnf}}_{\mu\nu}$, $\mathcal{PFOTL}\mathbf{1}_{\mu\nu}$, $\mathcal{PFOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}$ are closed under $\mathsf{cl}[\cdot]$.

Definition 71 (type, constant type)**.** A *type* for a $\mathcal{PFOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence ϕ is a subset t of

$$C_1[\phi] = \mathsf{cl}_1[\phi] \cup \big\{x = c, x \neq c \mid c \in \mathsf{con}[\phi]\big\}$$

that satisfies the following conditions.

(1) For any $\psi \in C_1[\phi]$, $\psi \in t \iff \mathsf{neg}[\phi] \notin t$.
(2) For any $(\psi_1 \wedge \psi_2) \in C_1[\phi]$, $(\psi_1 \wedge \psi_2) \in t \iff \psi_1 \in t$ and $\psi_2 \in t$.
(3) For any $(\psi_1 \vee \psi_2) \in C_1[\phi]$, $(\psi_1 \vee \psi_2) \in t \iff \psi_1 \in t$ or $\psi_2 \in t$.
(4) For any $\mu X\psi \in C_1[\phi]$, $\mu X\psi \in t \iff [\dddot{\psi}]\{\mu X\psi/X\} \in t$.
(5) For any $\nu X\psi \in C_1[\phi]$, $\nu X\psi \in t \iff [\dddot{\psi}]\{\nu X\psi/X\} \in t$.

We denote by $\mathsf{Types}(\phi)$ the set of all types for ϕ. Clearly, $\mathsf{Types}(\phi)$ is a subset of the powerset of $C_1[\phi]$. Since the size of $C_1[\phi]$ is linear in the size of ϕ, we infer that there are exponentially many types in the size of ϕ. A type t for ϕ is said to be a *constant type for c* if $x = c \in t$. We also say that t is a *constant type* if there is a constant c such that t is a constant type for c.

Definition 72 (state candidate, realizable state candidate)**.** Let ϕ be a $\mathcal{PFOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence. A *state candidate* for ϕ is a set of types for ϕ. We denote by $\mathsf{StateC}(\phi)$ the set of all state candidates for ϕ. There are doubly exponentially many of them in the size of ϕ. We say that the first-order structure $\mathfrak{D}$ over σ_{surr} *realizes* a state candidate T for ϕ if $T = \big\{t^{\mathfrak{D}}(a) \mid a \in \mathcal{D}\big\}$. A state candidate is said to be *(finitely) realizable* iff there exists a (finite) first-order structure that realizes it. We denote by $\mathsf{Real}(\phi)$ the set of all realizable state candidates for ϕ.

Let T be a state candidate for ϕ that is realized by $\mathfrak{D} = (\mathcal{D}, \cdot^{\mathfrak{D}})$. It is easy to see that for every constant type t in T, $\big|\mathfrak{D}_{[t]}\big| = 1$. Let c be a constant that appears in ϕ. Observe that $x = c$ belongs to exactly one type in T.

Definition 73 (state function, run)**.** Let ϕ be a $\mathcal{PFOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence. A *state function* for ϕ is a function that maps each $u \in \mathbb{N}$ to a realizable state candidate for ϕ. Let f be a state function for ϕ. A *run* r in f is a function in $\prod_{u\in\mathbb{N}} f(u)$ such that the following hold.

- For every $\bigcirc\psi \in C_1[\phi]$ and every $u \in \mathbb{N}$, $\bigcirc\psi \in r(u) \iff \psi \in r(u+1)$.
- For every $\bullet\psi \in C_1[\phi]$ and every $u \in \mathbb{N}$, $\bullet\psi \in r(u) \iff u > 0$ and $\psi \in r(u-1)$.
- For every $\ominus\psi \in C_1[\phi]$ and every $u \in \mathbb{N}$, $\ominus\psi \in r(u) \iff u = 0$ or $[u > 0$ and $\psi \in r(u-1)]$.
- For every constant $c \in \mathsf{con}[\phi]$ and every $u, v \in \mathbb{N}$, $x = c \in r(u) \iff x = c \in r(v)$.

We say that a run r is a *constant run for* c if $x = c \in r(u)$ for some (for all) $u \in \mathbb{N}$. A run r is a *constant run* if there is a constant c such that r is a constant run for c.

Notice that there is only one run that goes through a constant type. Let f be a state function and t be a type in $f(u)$ with $x = c$ in t. Assume for contradiction that there are runs r_1, r_2 such that $r_1(u) = r_2(u) = t$ and $r_1 \neq r_2$. This means that there is a moment $v \in \mathbb{N}$ such that $r_1(v) \neq r_2(v)$, $x = c \in r_1(v)$, and $x = c \in r_2(v)$. But, we have already established that $x = c$ belongs to exactly one type in $f(v)$. Contradiction. So, a constant run (say, for a constant c) follows all the types that contain $x = c$.

Another useful remark is that constant runs are essential for "holding the interpretations of constants together" across time. Let f be a state function and $\mathfrak{D}_u$ be a σ_{surr}-structure that realizes $f(u)$. Suppose that at moment u, the constants $c_1, \ldots, c_k$ are all interpreted as a single domain element, say $a \in \mathcal{D}_u$. Clearly, $t_u = t^{\mathfrak{D}_u}(a)$ contains $\{x = c_1, \ldots, x = c_k\}$. There is a run r such that $r(u) = t_u$. Let v be an arbitrary moment. By definition of runs, $x = c_i \in r(v)$ for all $i = 1, \ldots, k$. It follows that $c_1, \ldots, c_k$ are interpreted as a single domain element by $\mathfrak{D}_v$.

Definition 74 (quasimodel)**.** Let ϕ be a $\mathcal{PFOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}$-sentence, f be a state function for ϕ, and $\mathscr{R}$ be a set of runs in f. The pair $\mathfrak{m} = (f, \mathscr{R})$ is a *quasimodel* for ϕ if for any $u \in \mathbb{N}$ and any $t \in f(u)$, there is $r \in \mathscr{R}$ such that $r(u) = t$.

Theorem 75. Let ϕ be a $\mathcal{PFOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence. Then, ϕ is satisfiable if and only if there is a well-founded adorned quasimodel for ϕ, in which ϕ is satisfied.

Proof. The $\Rightarrow$ direction is straightforward. Conversely, suppose that ϕ is satisfied at moment w in some well-founded adorned quasimodel $\mathfrak{m} = (f, \mathscr{R}, \tau)$ for ϕ. Every quasistate $f(u)$ is realized by some σ_{surr}-structure $\mathfrak{D}_u = (\mathcal{D}_u, \cdot^{\mathfrak{D}_u})$. There are only finitely many realizable candidates, and hence we can assume that $\{\mathfrak{D}_u \mid u \in \mathbb{N}\}$ is finite. Let κ be a cardinal that exceeds the cardinality of every $\mathcal{D}_u$ and $\mathscr{R}_c$ be the set of constant runs in $\mathscr{R}$. We will construct a model with domain

$$\mathcal{E} = \mathscr{R}_c \cup \left[(\mathscr{R} \setminus \mathscr{R}_c) \times \kappa\right] .$$

Let $u \in \mathbb{N}$ and t be a type in $f(u)$. If t is a constant type, then $\mathfrak{D}_{u[t]} = \{a\}$ for some $a \in \mathcal{D}_u$. Define $R_{u,t} = \{(a, r)\}$, where r is the unique run that goes throught t at u. If t is not a constant type, then take any surjection $\pi_{u,t}$ from κ to $\mathfrak{D}_{u[t]}$ (κ is bigger than $\mathfrak{D}_{u[t]}$) and define

$$aR_{u,t}(r, \xi) \overset{\text{def}}{\iff} r(u) = t \text{ and } \pi_{u,t}(\xi) = a .$$

Let $R_u = \bigcup_{t\in f(u)} R_{u,t}$ and observe that $R_u \subseteq \mathcal{D}_u \times \mathcal{E}$ is a left-total, injective, and right-total relation.

We define a family of σ_{surr}-structures $\{\mathfrak{E}_u\}_{u\in\mathbb{N}}$ so that the conditions of Lemma 69 are satisfied. It follows that the quasistates are realized by structures that share the same domain. The constants are interpreted rigidly. The rest of the proof proceeds like in Theorem 56. $\square$

Theorem 76. Let ϕ be a $\mathcal{PFOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentence. Then, ϕ is finitely satisfiable if and only if there is a well-founded adorned finitary quasimodel for ϕ, in which ϕ is satisfied.

Proof. A trivial modification of the proof for Theorem 75. $\square$

Theorem 77.

- The (finite) satisfiability problem for $\mathcal{PFOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentences is complete for 2EXPTIME.

- The (finite) satisfiability problem for $\mathcal{PFOTL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}[\sigma]$-sentences with a bounded number of variables or with predicate symbols of bounded arity is complete for $\mathsf{EXPSPACE}$.

Proof. The automata-theoretic arguments of Section 7 transfer directly to the monodic packed case. The packed fragment of first-order logic is known to have the finite model property [27, 31]. Its satisfiability problem is complete for $\mathsf{2EXPTIME}$ [22]. In the bounded-variable or bounded-arity case it is complete for $\mathsf{EXPSPACE}$ [22]. □

9. Monodic Guarded Fragments with Time-Fixpoints and Domain-Fixpoints

The guarded fragment was introduced by Andréka, Németi, and van Benthem in [2] as a natural fragment of first-order logic that extends the modal fragment (the image of modal formulas through the standard translation) and inherits nice properties of modal logic, such as Beth definability, decidability, finite model property. It consists of the first-order formulas in which quantification is relativized appropriately to atomic formulas. The guarded fragment was generalized to the loosely guarded fragment by van Benthem so that quantifiers are appropriately relativized to conjunctions of atomic formulas [57]. Let us note that the standard translation of an Until formula falls outside the guarded fragment, but is in the loosely guarded fragment. Marx further extended the loosely guarded fragment to the packed fragment, in which the guards for the quantifiers are more general [41]. The clique-garded fragment of Grädel, introduced in [21], is a syntactic variant of the packed fragment.

The guarded fragments were introduced in an attempt to explain the good behaviour of modal logics, and indeed they turn out to have some nice properties, one of which is decidability within doubly exponential time. The extension of modal logic with fixpoint operators, namely the μ-calculus, is known to be decidable within exponential time [14, 15]. The same is also known for the extension of μ-calculus with backward modalities [60]. We are naturally led to ask the question of what happens if we extend the guarded fragments with fixpoint operators. Grädel and Walukiewicz addressed this question in [24], where it is shown the satisfiability problems for both the guarded fixpoint fragment (μGF) and the loosely guarded fixpoint fragment (μLGF) are complete for $\mathsf{2EXPTIME}$. Moreover, if we restrict the formulas to those that have a bounded number of variables, then the satisfiability problem becomes $\mathsf{EXPTIME}$-complete.

In this section we will consider monodic guarded fragments with fixpoint operators both on the time dimension (μ and ν) and on the domain dimension (only μ). We will extend the technique presented in Section 8 in order to get a “temporal model iff well-founded quasimodel” theorem. The automata-theoretic techniques used so far will then apply immediately and give us upper complexity bounds.

First, we will consider the extension of full first-order logic with least and greatest fixpoint operators and we will show a simple lemma for a satisfiability reduction class, in which fixpoint formulas are “guarded” by atomic formulas. We note that when placing these guards it is essential that the formula is in positive normal form.

Definition 78 (FO with equality + least and greatest fixpoints)**.** Let $\sigma = (\mathcal{P}, \mathcal{P}_0, \mathcal{F}, \mathcal{C}, \mathsf{ar})$ be a first-order signature, $\mathcal{V}$ be a countably infinite set of individual variables. We introduce a set of *relation variables* $\mathcal{Z} = \bigcup_{k\in\mathbb{N}} \mathcal{Z}_k$, where $\mathcal{Z}_k$ is a countably infinite set of k-ary relation variables for every $k \in \mathbb{N}$. We define $\mathcal{FOL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$ as the smallest set that satisfies the following conditions.

- $\mathcal{FOL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$ contains $\mathcal{P}_0$ and $\{\neg p \mid p \in \mathcal{P}_0\}$.
- If t_1, t_2 are σ-terms, then $t_1 = t_2$ and $t_1 \neq t_2$ are in $\mathcal{FOL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$.
- If P is a n-ary predicate symbol in $\mathcal{P}$ and $t_1, \ldots, t_n$ are σ-terms, then $P(t_1, \ldots, t_n)$ and $\neg P(t_1, \ldots, t_n)$ are in $\mathcal{FOL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$.
- If Z is a n-ary relation variable and $t_1, \ldots, t_n$ are σ-terms, then $Z(t_1, \ldots, t_n)$ and $\neg Z(t_1, \ldots, t_n)$ are in $\mathcal{FOL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$.
- $\mathcal{FOL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$ is closed under $\wedge$, $\vee$, existential and universal quantification.
- (*unguarded fixpoints*) If $\phi \in \mathcal{FOL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$, Z is a n-ary relation variable, $x_1, \ldots, x_n$ are individual variables, $\mathsf{fvars}[\phi] \subseteq \{x_1, \ldots, x_n\}$, and all free occurrences of Z in ϕ are positive, then $\mu Z x_1 \ldots x_n.\phi$ and $\nu Z x_1 \ldots x_n.\phi$ are in $\mathcal{FOL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$.

A *dom-fp-sentence* is a formula with no free relation variables. A *dom-sentence* is a formula with no free individual variables. A *sentence* is a formula that is both a dom-fp-sentence and a dom-sentence.

Definition 79 (truth)**.** Let $\mathfrak{D} = (\mathcal{D}, \cdot^{\mathfrak{D}})$ be a σ-structure, $h : \mathcal{V} \to \mathcal{D}$ be an individual variable assignment, and η be a *relation variable assignment*, i.e. a function that maps a relation variable Z to a n-ary relation on $\mathcal{D}$, where n is the arity of Z. We define the truth relation inductively as follows.

$$\begin{aligned}
\mathfrak{D}, h, \eta \models Z(t_1, \ldots, t_n) &\iff ([\![t_1]\!]_h^{\mathfrak{D}}, \ldots, [\![t_n]\!]_h^{\mathfrak{D}}) \in \eta(Z) \\
\mathfrak{D}, h, \eta \models \mu Z x_1 \ldots x_n.\phi &\iff \langle h(x_1), \ldots, h(x_n)\rangle \text{ is in the least fixpoint of } \phi_{h,\eta}^{\mathfrak{M}} \\
\mathfrak{D}, h, \eta \models \nu Z x_1 \ldots x_n.\phi &\iff \langle h(x_1), \ldots, h(x_n)\rangle \text{ is in the greatest fixpoint of } \phi_{h,\eta}^{\mathfrak{M}}
\end{aligned}$$

where $\phi_{h,\eta}^{\mathfrak{M}} : \wp(\mathcal{D}^n) \to \wp(\mathcal{D}^n)$ is defined as

$$\phi_{h,\eta}^{\mathfrak{M}}(S) = \left\{(a_1, \ldots, a_n) \in \mathcal{D}^n \mid \mathfrak{D}, \{(x_1, a_1), \ldots, (x_n, a_n)\}, \eta[Z \mapsto S] \models \phi\right\} .$$

Truth for the rest of the cases is defined in the obvious way. As in Proposition 4, we can show that semantics is well-defined.

Since free relation variables are exactly like predicate symbols as far as truth in concerned, for every $\mathcal{FOL}_{\mu\nu}^{\mathsf{pos}}[\sigma]$-formula we can easily construct an equisatisfiable $\mathcal{FOL}_{\mu\nu}^{\mathsf{pos}}[\sigma]$-dom-fp-sentence.

Definition 80 (guards for fixpoint formulas, $\mathcal{FOL}_{\mu\nu g}^{\mathsf{pos}}$)**.** Fix a first-order signature σ. We expand σ to σ_g, which contains additionally a new predicate symbol Q_ϕ for every $\mathcal{FOL}_{\mu\nu}^{\mathsf{pos}}[\sigma]$-formula ϕ of the form ${}^{\mu}_{\nu}Z x_1 \ldots x_n.\psi$. We define the function $\mathsf{guard}_{\mu\nu}$ inductively as follows.

$$\begin{aligned}
\mathsf{guard}_{\mu\nu}[\phi] &= \phi \text{ , for any atomic or negated atomic } \phi \\
\mathsf{guard}_{\mu\nu}[\phi_1 \otimes \phi_2] &= \mathsf{guard}_{\mu\nu}[\phi_1] \otimes \mathsf{guard}_{\mu\nu}[\phi_2] \text{ , for } \otimes \in \{\wedge, \vee\} \\
\mathsf{guard}_{\mu\nu}[Qx\phi] &= Qx\mathsf{guard}_{\mu\nu}[\phi] \text{ , for } Q \in \{\exists, \forall\} \\
\mathsf{guard}_{\mu\nu}[fZx_1 \ldots x_n.\phi] &= fZx_1 \ldots x_n.(Q_{fZx_1 \ldots x_n.\phi} x_1 \ldots x_n \wedge \mathsf{guard}_{\mu\nu}[\phi]) \text{ , for } f \in \{\mu, \nu\}
\end{aligned}$$

We define $\mathcal{FOL}_{\mu\nu g}^{\mathsf{pos}}[\sigma]$ (the g in the subscript stands for 'guarded fixpoint formulas') as the smallest set that satisfies the following.

- $\mathcal{FOL}_{\mu\nu g}^{\mathsf{pos}}[\sigma]$ contains the atomic formulas, including equalities, as well as their negations.
- $\mathcal{FOL}_{\mu\nu g}^{\mathsf{pos}}[\sigma]$ is closed under $\wedge, \vee$, existential and universal quantification.
- (*guarded fixpoints*) If $\phi \in \mathcal{FOL}_{\mu\nu g}^{\mathsf{pos}}[\sigma]$, Z is a n-ary relation variable, $x_1, \ldots, x_n$ are individual variables, $\mathsf{fvars}[\phi] \subseteq \{x_1, \ldots, x_n\}$, Q is a n-ary predicate symbol in σ, and all free occurrences of Z in ϕ are positive, then $\mu Z x_1 \ldots x_n.(Q x_1 \ldots x_n \wedge \phi)$ and $\nu Z x_1 \ldots x_n.(Q x_1 \ldots x_n \wedge \phi)$ are in $\mathcal{FOL}_{\mu\nu g}^{\mathsf{pos}}[\sigma]$.

It is clear that for any $\mathcal{FOL}_{\mu\nu}^{\mathsf{pos}}[\sigma]$-formula ϕ, $\mathsf{guard}_{\mu\nu}[\phi]$ is in $\mathcal{FOL}_{\mu\nu g}^{\mathsf{pos}}[\sigma_g]$.

Claim 81. Let ϕ be a $\mathcal{FOL}_{\mu\nu}^{\mathsf{pos}}[\sigma]$-formula, and $\mathfrak{D} = (\mathcal{D}, \cdot^{\mathfrak{D}})$ be a σ-structure. We define the σ_g-structure $\mathfrak{E} = (\mathcal{D}, \cdot^{\mathfrak{E}})$, where $\cdot^{\mathfrak{E}}$ extends $\cdot^{\mathfrak{D}}$ as $Q_\phi^{\mathfrak{E}} = \mathcal{D}^n$ for any ϕ of the form ${}^{\mu}_{\nu}Z x_1 \ldots x_n.\psi$. Then, for any individual variable assignment h, and any relation variable assignment η,

$$\mathfrak{D}, h, \eta \models \phi \implies \mathfrak{E}, h, \eta \models \mathsf{guard}_{\mu\nu}[\phi] .$$

Proof. By induction on ϕ.

- The atomic cases $p, t_1 = t_2, P(t_1, \ldots, t_n), Z(t_1, \ldots, t_n)$ as well as the negated atomic cases $\neg p, t_1 \neq t_2, \neg P(t_1, \ldots, t_n), \neg Z(t_1, \ldots, t_n)$ are all straightforward.
- The cases $\phi = (\phi_1 \wedge \phi_2), (\phi_1 \vee \phi_2), \exists x\phi_1, \forall x\phi_1$ are easy.
- $\phi = \mu Z x_1 \ldots x_n.\phi_1$. Fix $\mathfrak{D}$ and define $\mathfrak{E}$ as described. Fix h, η and suppose that $\mathfrak{D}, h, \eta \models \phi$. We have that

$$\mathsf{guard}_{\mu\nu}[\mu Z x_1 \ldots x_n.\phi_1] = \mu Z x_1 \ldots x_n.(Q_\phi x_1 \ldots x_n \wedge \mathsf{guard}_{\mu\nu}[\phi_1]) .$$

Define $\bar{x} = (x_1, \ldots, x_n)$, $\bar{a} = (a_1, \ldots, a_n)$, $\bar{x} \mapsto \bar{a} = \{(x_1, a_1), \ldots, (x_n, a_n)\}$, and the functions

$$\begin{aligned}
f_1(S) &= \left\{\bar{a} \in \mathcal{D}^n \mid \mathfrak{D}, \bar{x} \mapsto \bar{a}, \eta[Z \mapsto S] \models \phi_1\right\} \\
f_2(S) &= \left\{\bar{a} \in \mathcal{D}^n \mid \mathfrak{E}, \bar{x} \mapsto \bar{a}, \eta[Z \mapsto S] \models Q_\phi x_1 \ldots x_n \wedge \mathsf{guard}_{\mu\nu}[\phi_1]\right\} \\
&= Q_\phi^{\mathfrak{E}} \cap \left\{\bar{a} \in \mathcal{D}^n \mid \mathfrak{E}, \bar{x} \mapsto \bar{a}, \eta[Z \mapsto S] \models \mathsf{guard}_{\mu\nu}[\phi_1]\right\} \\
&= \left\{\bar{a} \in \mathcal{D}^n \mid \mathfrak{E}, \bar{x} \mapsto \bar{a}, \eta[Z \mapsto S] \models \mathsf{guard}_{\mu\nu}[\phi_1]\right\}
\end{aligned}$$

The inductive hypothesis gives us that $f_1(S) \subseteq f_2(S)$ for all $S \subseteq \mathcal{D}^n$, which implies that $\mathit{lfp}(f_1) \subseteq \mathit{lfp}(f_2)$. From $\langle h(x_1), \ldots, h(x_n)\rangle \in \mathit{lfp}(f_1)$, we get that $\langle h(x_1), \ldots, h(x_n)\rangle \in \mathit{lfp}(f_2)$ and hence $\mathfrak{E}, h, \eta \models \mathsf{guard}_{\mu\nu}[\phi]$.

- The case $\phi = \nu Z x_1 \ldots x_n.\phi_1$ is similar to the previous one. $\square$

Claim 82. Let ϕ be a $\mathcal{FOL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$-formula. For any σ_g-structure $\mathfrak{D} = (\mathcal{D}, \cdot^{\mathfrak{D}})$, any individual variable assignment h, and any relation variable assignment η,

$$\mathfrak{D}, h, \eta \models \mathsf{guard}_{\mu\nu}[\phi] \implies \mathfrak{D}, h, \eta \models \phi\ .$$

Proof. By induction on ϕ.

- The base cases are trivial and the cases $\phi = (\phi_1 \wedge \phi_2), (\phi_1 \vee \phi_2), \exists x\phi_1, \forall x\phi_1$ are easy.
- $\phi = \mu Z x_1 \ldots x_n.\phi_1$. Fix $\mathfrak{D}, h, \eta$ and suppose that

$$\mathfrak{D}, h, \eta \models \mathsf{guard}_{\mu\nu}[\phi] = \mu Z x_1 \ldots x_n.(Q_\phi x_1 \ldots x_n \wedge \mathsf{guard}_{\mu\nu}[\phi_1])\ .$$

We define the funtions

$$\begin{aligned} f_1(S) &= \big\{\bar{a} \in \mathcal{D}^n \mid \mathfrak{D}, \bar{x} \mapsto \bar{a}, \eta[Z \mapsto S] \models Q_\phi x_1 \ldots x_n \wedge \mathsf{guard}_{\mu\nu}[\phi_1]\big\} \\ &= Q_\phi^{\mathfrak{D}} \cap \underbrace{\big\{\bar{a} \in \mathcal{D}^n \mid \mathfrak{D}, \bar{x} \mapsto \bar{a}, \eta[Z \mapsto S] \models \mathsf{guard}_{\mu\nu}[\phi_1]\big\}}_{f_1'(S)} \\ f_2(S) &= \big\{\bar{a} \in \mathcal{D}^n \mid \mathfrak{D}, \bar{x} \mapsto \bar{a}, \eta[Z \mapsto S] \models \phi_1\big\} \end{aligned}$$

and observe that, by the inductive hypothesis, $f_1'(S) \subseteq f_2(S)$ for all $S \subseteq \mathcal{D}^n$. It follows that $f_1(S) \subseteq f_2(S)$ for all $S \subseteq \mathcal{D}^n$. Hence, $\mathit{lfp}(f_1) \subseteq \mathit{lfp}(f_2)$ and we are done.

- The case $\phi = \nu Z x_1 \ldots x_n.\phi_1$ is dealt with similar arguments. $\square$

Lemma 83. Let ϕ be a $\mathcal{FOL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$-formula. Then, ϕ is equisatisfiable to $\mathsf{guard}_{\mu\nu}[\phi]$.

Proof. An immediate consequence of Claim 81 and Claim 82. $\square$

The idea behind the introduction of these guards for fixpoint formulas is that they will allow us to extend Lemma 69 when we add to the packed fragment "guarded" least fixpoint formulas. In the case of greatest fixpoints, this technique does not work. Similarly to the packing guards, these least fixpoint guards will enforce the R-thinness of the domain tuples in the least fixpoints.

Definition 84 (packed fragment + unguarded least fixpoints, packed fragment + guarded least fixpoints)**.** Fix a first-order signature σ. We define $\mathcal{PFOL}^{\mathsf{pos}}_{\mu}[\sigma]$ as the smallest set that satisfies the following.

- $\mathcal{PFOL}^{\mathsf{pos}}_{\mu}[\sigma]$ contains atomic formulas, including equalities, as well as their negations.
- $\mathcal{PFOL}^{\mathsf{pos}}_{\mu}[\sigma]$ is closed $\wedge, \vee$, existential packed quantification, and universal packed quantification.
- (*unguarded least fixpoints*) If $\phi \in \mathcal{PFOL}^{\mathsf{pos}}_{\mu}[\sigma]$, Z is a n-ary relation variable, $x_1, \ldots, x_n$ are individual variables, $\mathsf{fvars}[\phi] \subseteq \{x_1, \ldots, x_n\}$, and all free occurrences of Z in ϕ are positive, then $\mu Z x_1 \ldots x_n.\phi$ is in $\mathcal{PFOL}^{\mathsf{pos}}_{\mu}[\sigma]$.

$\mathcal{PFOL}^{\mathsf{pos}}_{\mu g}[\sigma]$ is defined as the smallest set that satisfies the following.

- $\mathcal{PFOL}^{\mathsf{pos}}_{\mu g}[\sigma]$ contains atomic formulas, including equalities, as well as their negations.
- $\mathcal{PFOL}^{\mathsf{pos}}_{\mu g}[\sigma]$ is closed $\wedge, \vee$, existential packed quantification, and universal packed quantification.
- (*guarded least fixpoints*) If $\phi \in \mathcal{PFOL}^{\mathsf{pos}}_{\mu g}[\sigma]$, Z is a n-ary relation variable, $x_1, \ldots, x_n$ are individual variables, $\mathsf{fvars}[\phi] \subseteq \{x_1, \ldots, x_n\}$, Q is a n-ary predicate symbol, and all free occurrences of Z in ϕ are positive, then $\mu Z x_1 \ldots x_n.(Q x_1 \ldots x_n \wedge \phi)$ is in $\mathcal{PFOL}^{\mathsf{pos}}_{\mu g}[\sigma]$.

It is obvious that for any $\phi \in \mathcal{PFOL}^{\mathsf{pos}}_{\mu}[\sigma]$, $\mathsf{guard}_{\mu\nu}[\phi] \in \mathcal{PFOL}^{\mathsf{pos}}_{\mu g}[\sigma]$.

Lemma 85. Let $\mathfrak{D} = (\mathcal{D}, \cdot^{\mathfrak{D}})$, $\mathfrak{E} = (\mathcal{E}, \cdot^{\mathfrak{E}})$ be σ-structures and $R \subseteq \mathcal{D} \times \mathcal{E}$ be a left-total, injective, and right-total relation. Assume that the conditions (ii), (iii), (iv) of Lemma 54, and the conditions (v), (vi) of Claim 67 hold. We replace condition (i) of Lemma 54 by condition (vii) of Lemma 69. For any $\mathcal{PFOL}^{\mathsf{pos}}_{\mu g}[\sigma]$-formula ϕ, any individual variable assignments h :

$\mathsf{fvars}[\phi] \to \mathcal{D}$, $g : \mathsf{fvars}[\phi] \to \mathcal{E}$, with hRg and g R-thin, and any relation variable assignments $\eta : \mathcal{Z} \to \bigcup_{k\in\mathbb{N}} \wp\left(\mathcal{D}^k\right)$ and $\theta : \mathcal{Z} \to \bigcup_{k\in\mathbb{N}} \wp\left(\mathcal{E}^k\right)$ with $\eta R_t \theta$,

$$\mathfrak{D}, h, \eta \models \phi \iff \mathfrak{E}, g, \theta \models \phi \ .$$

We write $\eta R_t \theta$ to mean that for any n-ary relation variable Z, $\theta(Z) = R_t(\eta(Z))$. For $S \subseteq \mathcal{D}^n$, we define

$$R_t(S) \stackrel{\text{def}}{=} \left\{(b_1, \dots, b_n) \in \mathcal{E}^n \mid \{b_1, \dots, b_n\} \text{ is } R\text{-thin } \& \ \langle R^{-1}(b_1), \dots, R^{-1}(b_n)\rangle \text{ is in } S\right\} \ .$$

We will also say that $\bar{b} = (b_1, \dots, b_n)$ is R-thin, whenever $\{b_1, \dots, b_n\}$ is R-thin and that $\bar{a}R\bar{b}$, whenever $a_i R b_i$ for all $i = 1, \dots, n$.

Proof. Fix $\mathfrak{D}, \mathfrak{E}, R$. The proof proceeds by induction on ϕ.

- The atomic cases, including the case $Z(t_1, \dots, t_n)$, are handled as in Lemma 69. The property follows immediately for the negated atomic cases.
- The cases $\phi = (\phi_1 \wedge \phi_2), (\phi_1 \vee \phi_2)$ are easy.
- For the cases of existential and universal packed quantification, we argue as in Lemma 69.
- $\phi = \mu Z x_1 \dots x_n.(Q x_1 \dots x_n \wedge \phi_1)$. Fix h, g with hRg, g R-thin and η, θ with $\eta R_t \theta$. Define

$$f_1(S) = \left\{\bar{a} \in \mathcal{D}^n \mid \mathfrak{D}, \bar{x} \mapsto \bar{a}, \eta[Z \mapsto S] \models Q x_1 \dots x_n \wedge \phi_1\right\}$$
$$f_2(S) = \left\{\bar{b} \in \mathcal{E}^n \mid \mathfrak{E}, \bar{x} \mapsto \bar{b}, \theta[Z \mapsto S] \models Q x_1 \dots x_n \wedge \phi_1\right\}$$

Let S be an arbitrary subset of $\mathcal{D}^n$. We argue that $R_t(f_1(S)) = f_2(R_t(S))$. Let $\bar{b} \in R_t(f_1(S))$. Then, $\bar{b}$ is R-thin and $\bar{a} = \langle R^{-1}(b_1), \dots, R^{-1}(b_n)\rangle$ is in $f_1(S)$. It follows that

$$\mathfrak{D}, \bar{x} \mapsto \bar{a}, \eta[Z \mapsto S] \models Q x_1 \dots x_n \wedge \phi_1 \ ,$$

which implies that $\bar{a} \in Q^{\mathfrak{D}}$ and $\mathfrak{D}, \bar{x} \mapsto \bar{a}, \eta[Z \mapsto S] \models \phi_1$. By definition of $\cdot^{\mathfrak{E}}$, $\bar{b} \in Q^{\mathfrak{E}}$ and hence $\mathfrak{E}, \bar{x} \mapsto \bar{b}, \theta[Z \mapsto R_t(S)] \models Q x_1 \dots x_n$. Observe that $(\bar{x} \mapsto \bar{a})R(\bar{x} \mapsto \bar{b})$, $(\bar{x} \mapsto \bar{b})$ is R-thin, and $\eta[Z \mapsto S]R_t\theta[Z \mapsto R_t(S)]$. By the inductive hypothesis, $\mathfrak{E}, \bar{x} \mapsto \bar{b}, \theta[Z \mapsto R_t(S)] \models \phi_1$. We deduce that $\bar{b} \in f_2(R_t(S))$. For the converse, suppose that $\bar{b} \in f_2(R_t(S))$. We have that

$$\mathfrak{E}, \bar{x} \mapsto \bar{b}, \theta[Z \mapsto R_t(S)] \models Q x_1 \dots x_n \wedge \phi_1 \ ,$$

from which it follows that $\bar{b} \in Q^{\mathfrak{E}}$ and $\mathfrak{E}, \bar{x} \mapsto \bar{b}, \theta[Z \mapsto R_t(S)] \models \phi_1$. Notice that $\bar{b}$ is R-thin and that $\bar{a} \in Q^{\mathfrak{D}}$, where $\bar{a} = \langle R^{-1}(b_1), \dots, R^{-1}(b_n)\rangle$. The inductive hypothesis gives us that $\mathfrak{D}, \bar{x} \mapsto \bar{a}, \eta[Z \mapsto S] \models \phi_1$ and hence $\bar{a} \in f_1(S)$. Immediately, we get that $\bar{b} \in R_t(f_1(S))$.

By an easy well-founded induction, we infer that $R_t(\mu_\alpha(f_1)) = \mu_\alpha(f_2)$ for any ordinal α. Basis: $R_t(\mu_0(f_1)) = R_t(\emptyset) = \emptyset = \mu_0(f_2)$. Step:

$$R_t(\mu_\alpha(f_1)) = R_t\left(\bigcup_{\beta<\alpha} \mu_\beta(f_1)\right) = \bigcup_{\beta<\alpha} R_t(\mu_\beta(f_1)) \stackrel{\text{ind. hyp.}}{=} \bigcup_{\beta<\alpha} \mu_\beta(f_2) = \mu_\alpha(f_2) \ .$$

As an immediate consequence, we get that

$$R_t(\mathit{lfp}(f_1)) = R_t\left(\bigcup_{\alpha\in\mathbf{Ord}} \mu_\alpha(f_1)\right) = \bigcup_{\alpha\in\mathbf{Ord}} R_t(\mu_\alpha(f_1)) = \bigcup_{\alpha\in\mathbf{Ord}} \mu_\alpha(f_2) = \mathit{lfp}(f_2) \ .$$

Let $\bar{a} = \langle h(x_1), \dots, h(x_n)\rangle$ and $\bar{b} = \langle g(x_1), \dots, g(x_n)\rangle$. From R-thinness of g, we get that $\bar{b}$ is R-thin. Suppose that $\mathfrak{D}, h, \eta \models \phi$. Then, $\bar{a} \in \mathit{lfp}(f_1)$ and hence $\bar{b} \in R_t(\mathit{lfp}(f_1)) = \mathit{lfp}(f_2)$. It follows that $\mathfrak{E}, g, \theta \models \phi$. Conversely, suppose that $\mathfrak{E}, g, \theta \models \phi$. We get that $\bar{b} \in \mathit{lfp}(f_2) = R_t(\mathit{lfp}(f_1))$ and therefore $\bar{a} \in \mathit{lfp}(f_1)$. This means that $\mathfrak{D}, h, \eta \models \phi$. ☐

Remark 86 (what goes wrong with greatest fixpoints?)**.** Let us try to see where exactly the proof breaks down if we consider guarded greatest fixpoints as well. The induction of Lemma 85 would include the case $\phi = \nu Z x_1 \dots x_n.(Q x_1 \dots x_n \wedge \phi_1)$.

First, we observe that the function R_t is injective. Let S_1, S_2 be two arbitrary subsets of $\mathcal{D}^n$ with $R_t(S_1) = R_t(S_2)$. Let $\bar{a} = (a_1, \dots, a_n)$ be an arbitrary element of S_1. Among $a_1, \dots, a_n$ there may be some elements repeating. For any $a \in \{a_1, \dots, a_n\}$ take $b \in \mathcal{E}$ such that aRb (there is at least one such element, since R is left-total) and put $f(a) = b$. Define

$\bar{b} = (b_1, \ldots, b_n) = \langle f(a_1), \ldots, f(a_n) \rangle$ and notice that $\bar{b}$ is R-thin and that $\bar{a}R\bar{b}$. So, $\bar{b}$ is in $R_t(S_1)$ and hence in $R_t(S_2)$. It follows that $\bar{a}$ is in S_2. We have established that $S_1 \subseteq S_2$. By symmetry, $S_1 = S_2$. Since R_t is injective, we have the property $R_t(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} R_t(A_i)$.

Fix h, g with hRg, g R-thin and η, θ with $\eta R_t \theta$. Define $f_1(S), f_2(S)$ as in the least fixpoint case. We have that $R_t(f_1(S)) = f_2(R_t(S))$ for any $S \subseteq \mathcal{D}^n$. We argue now that for any ordinal α, $R_t(\nu_\alpha(f_1)) = \nu_\alpha(f_2)$. Every works fine for the induction step:

$$R_t(\nu_\alpha(f_1)) = R_t\left(\bigcap_{\beta<\alpha} \nu_\beta(f_1)\right) = \bigcap_{\beta<\alpha} R_t(\nu_\beta(f_1)) \overset{\text{ind. hyp.}}{=} \bigcap_{\beta<\alpha} \nu_\beta(f_2) = \nu_\alpha(f_2)\ .$$

The argument collapses for the base case: $R_t(\nu_0(f_1)) = R_t(\mathcal{D}^n)$ is not necessarily equal to $\nu_0(f_2) = \mathcal{E}^n$. The only thing we can show is that $\mathit{gfp}(f_1) \subseteq \mathit{gfp}(f_2)$.

Now we are ready to introduce the formulas of 'packed first-order logic with least fixpoints + temporal logic with least and greatest fixpoints', namely $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}$. We also consider a reduction class for satisfiability of $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}$-formulas, i.e. the formulas of 'packed first-order logic with guarded least fixpoints + temporal logic with least and greatest fixpoints', namely $\mathcal{PFOL}_{\mu g}\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}$.

Definition 87 ($\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}$, $\mathcal{PFOL}_{\mu g}\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}$)**.** Fix a first-order signature $\sigma = (\mathcal{P}, \mathcal{P}_0, \mathcal{F} = \emptyset, \mathcal{C}, \mathsf{ar})$ with no function symbols. We define $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$ as the smallest set that satisfies the following.

- $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$ includes $\mathcal{P}_0$, $\{\neg p \mid p \in \mathcal{P}_0\}$, the set of (temporal) fixpoint variables $\mathcal{X}$, and $\{\neg X \mid X \in \mathcal{X}\}$.
- If t_1, t_2 are σ-terms, then $t_1 = t_2$ and $t_1 \neq t_2$ are in $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$.
- For any n-ary predicate symbol P in σ, if $t_1, \ldots, t_n$ are σ-terms, then $P(t_1, \ldots, t_n)$ and $\neg P(t_1, \ldots, t_n)$ are in $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$.
- For any n-ary relation fixpoint variable $Z \in \mathcal{Z}$, if $t_1, \ldots, t_n$ are σ-terms, then $Z(t_1, \ldots, t_n)$ and $\neg Z(t_1, \ldots, t_n)$ are in $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$.
- $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$ is closed under $\wedge$, $\vee$, existential packed quantification, and universal packed quantification.
- (*unguarded domain-side least fixpoints*) If $\phi \in \mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$, Z is a n-ary relation variable, $x_1, \ldots, x_n$ are individual variables, $\mathsf{fvars}[\phi] \subseteq \{x_1, \ldots, x_n\}$, and all free occurrences of Z in ϕ are positive, then $\mu Z x_1 \ldots x_n.\phi$ is in $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$.
- $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}$ is closed under the temporal connectives $\bigcirc, \bullet, \ominus$.
- (*temporal fixpoints*) If $\phi \in \mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$, X is a (temporal) fixpoint variable, and all free occurrences of X in ϕ are positive, then $\mu X \phi$ and $\nu X \phi$ are in $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$.

$\mathcal{PFOL}_{\mu g}\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$ is defined similarly. Just replace the 'unguarded domain-side least fixpoints' rule by the following.

- (*guarded domain-side least fixpoints*) If $\phi \in \mathcal{PFOL}_{\mu g}\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$, Z is a n-ary relation variable, $x_1, \ldots, x_n$ are individual variables, $\mathsf{fvars}[\phi] \subseteq \{x_1, \ldots, x_n\}$, Q is a n-ary predicate symbol in σ, and all free occurrences of Z in ϕ are positive, then $\mu Z x_1 \ldots x_n.(Q x_1 \ldots x_n \wedge \phi)$ is in $\mathcal{PFOL}_{\mu g}\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}[\sigma]$.

It is clear that $\mathcal{PFOL}_{\mu g}\mathcal{TL}^{\mathsf{pos}}_{\mu\nu} \subseteq \mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}$. Semantics is defined as one would expect. Following our usual convention, we denote by $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pnf}}_{\mu\nu}$ the set of $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}$-formulas in positive normal form and by $\mathcal{PFOL}_{\mu g}\mathcal{TL}^{\mathsf{pnf}}_{\mu\nu}$ the set of $\mathcal{PFOL}_{\mu g}\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}$-formulas in positive normal form. Lemma 83 easily extends to $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}$, which means that $\mathcal{PFOL}_{\mu g}\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}$ is a reduction class for satisfiability of $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}$-formulas.

Since we have now three different types of variables, it is useful to introduce some more terminology for sentences. A *dom-sentence* is a formula with no free individual variables. A *dom-fp-sentence* is a formula with no free relation variables. A *time-fp-sentence* is a formula with no free temporal fixpoint variables. A *sentence* is a formula with no free variables of any type. It should be obvious how, given a formula, we can construct an equisatisfiable sentence.

Definition 88 (monodicity, $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mathbf{1}\mu\nu}$, $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mathbf{1}\mu\nu}$)**.** We will strengthen the monodicity requirement, so that domain-fixpoint formulas are purely non-temporal. The idea is that the evaluation of a domain-fixpoint is done entirely in the structure of a single moment. So, the set of monodic $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}$-formulas, $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mathbf{1}\mu\nu}$, is defined as the subset of $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}$ that satisfies the usual monodicity conditions and additionally

- (*domain-fixpoint monodicity requirement*) Any domain-fixpoint subformula $\mu Z x_1 \ldots x_n.\phi$ is purely non-temporal, i.e. ϕ contains no temporal operators and no temporal fixpoint variables.

Similarly, we define the set of monodic $\mathcal{PFOL}_{\mu g}\mathcal{TL}^{\mathsf{pos}}_{\mu\nu}$-formulas, $\mathcal{PFOL}_{\mu g}\mathcal{TL}^{\mathsf{pos}}_{\mathbf{1}\mu\nu}$. The sets of formulas $\mathcal{PFOL}_\mu\mathcal{TL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}$, $\mathcal{PFOL}_{\mu g}\mathcal{TL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}$ are defined in the obvious way.

We start with a $\mathcal{PFOL}_{\mu g}\mathcal{TL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}$-sentence ϕ that contains no function symbols. Extending the analysis of Section 8 is rather straightforward. Types, state candidates, runs, state functions, and quasimodels are defined in the same way. We get the "temporal model iff well-founded quasimodel" theorem using the same construction as in Theorem 75 by virtue of Lemma 85. Just observe that monodicity ensures that domain-side fixpoint formulas are treated as base cases in the crucial induction that establishes that formula occurrences in the quasimodel are satisfied at the respective moment and domain element.

Theorem 89. Let ϕ be a $\mathcal{PFOL}_{\mu g}\mathcal{TL}^{\mathsf{pnf}}_{\mathbf{1}\mu\nu}$-sentence. Then, ϕ is (finitely) satisfiable if and only if there is a well-founded adorned (finitary) quasimodel for ϕ, in which ϕ is satisfied.

Theorem 90.

(1) The satisfiability problem for monodic 'guarded first-order logic with least fixpoints + temporal logic with least and greatest fixpoints' over the naturals is complete for $\mathsf{2EXPTIME}$.
(2) The satisfiability problem for monodic 'loosely guarded first-order logic with least fixpoints + temporal logic with least and greatest fixpoints' over the naturals is complete for $\mathsf{2EXPTIME}$.
(3) The satisfiability problem for monodic 'bounded-variable or bounded-arity guarded first-order logic with least fixpoints + temporal logic with least and greatest fixpoints' over the naturals is complete for $\mathsf{EXPSPACE}$.
(4) The satisfiability problem for monodic 'bounded-variable or bounded-arity loosely guarded first-order logic with least fixpoints + temporal logic with least and greatest fixpoints' over the naturals is complete for $\mathsf{EXPSPACE}$.

Proof. The automata-theoretic arguments of Section 7.1 with trivial modifications apply here as well. We will only discuss (1). The rest of the claims are shown similarly. The non-temporal part is contained in the guarded fixpoint logic (with least and greatest fixpoints) of Grädel and Walukiewicz [24], which is in $\mathsf{2EXPTIME}$. We proceed to argue as in Theorem 63. Completeness is an immediate consequence of the fact that even guarded first-order logic (with no fixpoints) is complete for $\mathsf{2EXPTIME}$. $\square$

The most important question we leave here open is whether greatest fixpoints can be handled. The technique we use in Lemma 85 is based on placing atomic guards at least fixpoint formulas in order to enforce thinness of the fixpoints. It does not seem to extend at all to greatest fixpoints, since they are "evaluated from above". That is, the first evaluation involves all possible tuples over the domain and hence we cannot control which tuples eventually end up in the fixpoint.

It would also be interesting to examine if we can relax the monodicity restriction on domain-fixpoints. Remember that the regeneration of temporal fixpoints is restricted on a single run, by requiring that $\exists/\forall$-subformulas have no free temporal fixpoint variables. The analogue of this for domain-fixpoints would be that temporal $\bigcirc/\bullet/\ominus/\mu/\nu$-subformulas have no free relation variables. Thus, a domain-fixpoint is regenerated at the same moment even though it may have some temporal operators in it. We leave open the question of whether this weaker monodicity condition allows us to obtain similar results.

10. Applications on Temporal Description Logics

The field of knowledge representation is concerned with methods for describing a "domain of discourse" (an "application domain") as well as for designing intelligent systems that are able to find implicit consequences of the explicitly described knowledge. Description logics emerged in the context of network-based knowledge representation systems after the realization that semantics can be given to network-based representations, such as semantic networks and frames, using first-order logic. For these specialized structures, however, only certain fragments of first-order logic are required, which are more computationally manageable.

The notion of concepts is central in the study of description logics. Informally, a concept is a property held by some individuals of the domain. Alternatively, a concept can be viewed as the set of the invididuals that hold it. There are also relationships between individuals, which are modeled as binary relations. For example, the relation *isMotherOf* relates two individuals x, y if x is the mother of y. With the language of description logics we can express statements such as 'if x is a mother, then x is a woman' and 'if x is the mother of y, then y is a child of x'. A terminology is a collection of statements of this sort, that defines the concepts of the domain we want to model. Assertions for specific individuals are also considered, such as 'Mary is a mother', 'John is the grandfather of George'. A terminology together with a collection of assertions for individuals is called knowledge base.

Several reasoning tasks are of interest. The basic inference on concepts is subsumption checking, that is 'is concept D more general than concept C?'. For example, 'is every woman a mother?'. Subsumption is trivially reducible to satisfiability, as will be made apparent later, when satisfiability is defined. An important reasoning task for knowledge bases is consistency checking, which informally is about determining whether the information in the knowledge base is self-conflicting.

Temporal description logics have arisen in an effort to incorporate time in the formalisms of description logics. There is a bewildering variety of available choices when designing a temporal description logic. We refer the reader to the surveys [3, 40]. Our interest in temporal description logics stems from the fact that many of these logics can be embedded in the monodic fragments studied here. Thus, decidability and complexity results can be obtained directly as corollaries. In [16] an EXPSPACE-completeness result for a temporal description logic is claimed, but given without a proof.

We begin by introducing a simple (non-temporal) description logic, called $\mathcal{ALC}$, in order to prepare the way for the more involved setting of temporal description logics, which are presented immediately after. We conclude with a reduction of the (finite) satisfiability problem for a temporal description logic with fixpoints to the (finite) satisfiability problem for a monodic fragment. We apply our results to obtain tight complexity results.

10.1. **A Simple Description Logic.** We present the syntax and semantics of $\mathcal{ALC}$. Fix a set $N_C = \{A_0, A_1, \ldots\}$ of *atomic concepts* and a set $N_R = \{r_0, r_1, \ldots\}$ of *atomic roles*. Define the alphabet $\Sigma = N_C \cup N_R \cup \{\neg, \sqcap, \exists, (,)\}$.

Definition 91 ($\mathcal{ALC}$)**.** $\mathcal{ALC}$ is defined as the smallest subset of $\Sigma^\star$ that satisfies the following conditions.

- $\mathcal{ALC}$ contains N_C.
- If $C \in \mathcal{ALC}$, then $\neg C \in \mathcal{ALC}$.
- If $C_1, C_2 \in \mathcal{ALC}$, then $(C_1 \sqcap C_2) \in \mathcal{ALC}$.
- If $C \in \mathcal{ALC}$ and $r \in N_R$, then $\exists r.C \in \mathcal{ALC}$.

When we want to make explicit the sets of atomic concepts and atomic roles, we will write $\mathcal{ALC}[N_C, N_R]$ instead of $\mathcal{ALC}$. An element of $\mathcal{ALC}$ is called an $\mathcal{ALC}$*-concept* (or just *concept*). We will use the usual parenthesis elimination convetions when no confusion arises.

Definition 92 ($\mathcal{ALC}$ semantics)**.** We define semantics over an *interpretation* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, where $\Delta^\mathcal{I}$ is a nonempty set called the *domain* of $\mathcal{I}$, and $\cdot^\mathcal{I}$ is a function, called the *interpretation function*, that maps each atomic concept A to a subset $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$ and each atomic role r to a subset $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. We extend $\cdot^{\mathcal{I}}$ to $[\cdot]^{\mathcal{I}}$ inductively as follows.

$$[A]^{\mathcal{I}} = A^{\mathcal{I}} \qquad [\neg C]^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus [C]^{\mathcal{I}}$$
$$[r]^{\mathcal{I}} = r^{\mathcal{I}} \quad [C_1 \sqcap C_2]^{\mathcal{I}} = [C_1]^{\mathcal{I}} \cap [C_2]^{\mathcal{I}}$$
$$[\exists r.C]^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \text{there is } y \in \Delta^{\mathcal{I}} \text{ s.t. } (x,y) \in r^{\mathcal{I}} \text{ and } y \in [C]^{\mathcal{I}}\}$$

We say that an interpretation $\mathcal{I}$ *satisfies* an $\mathcal{ALC}$-concept C or that $\mathcal{I}$ is a *model* of C if $[C]^{\mathcal{I}} \neq \emptyset$. An $\mathcal{ALC}$-concept C is *satisfiable* if there is an interpretation that satisfies it.

We define the abbreviations $\bot = (C \sqcap \neg C)$, $\top = \neg\bot$, $(C_1 \sqcup C_2) = \neg(\neg C_1 \sqcap \neg C_2)$ and $\forall r.C = \neg\exists r.\neg C$. It is easy to see that

$$[\bot]^{\mathcal{I}} = \emptyset \quad [\top]^{\mathcal{I}} = \Delta^{\mathcal{I}}$$
$$[C_1 \sqcup C_2]^{\mathcal{I}} = [C_1]^{\mathcal{I}} \cup [C_2]^{\mathcal{I}}$$
$$[\forall r.C]^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \text{for any } y \in \Delta^{\mathcal{I}},\ (x,y) \in r^{\mathcal{I}} \implies y \in [C]^{\mathcal{I}}\} \ .$$

Definition 93 ($\mathcal{ALC}$-TBox)**.** An *$\mathcal{ALC}$-concept inclusion axiom* is a pair $C \dot{\sqsubseteq} D$, where C, D are $\mathcal{ALC}$-concepts. An $\mathcal{ALC}$-TBox is a finite set of $\mathcal{ALC}$-concept inclusion axioms.

An intepretation $\mathcal{I}$ is said to *satisfy* a concept inclusion axiom if $[C]^{\mathcal{I}} \subseteq [D]^{\mathcal{I}}$. An interpretation $\mathcal{I}$ *satisfies* (is a *model* of) a TBox if it satisfies all the concept inclusion axioms in the TBox. A TBox is *satisfiable* if there is an interpretation that satisfies it.

Definition 94 ($\mathcal{ALC}$-ABox)**.** Fix a set N_I of *individual names*. We extend the interpretation functions so that they map each individual name to an element of the domain. An *$\mathcal{ALC}$-concept assertion* is a pair $C(a)$, where C is an $\mathcal{ALC}$-concept and a is an individual name. A *role assertion* is a pair $r(a, b)$, where r is a role and a, b are individual names. An $\mathcal{ALC}$-ABox is a finite set of $\mathcal{ALC}$-concept and role assertions.

We say that an interpretation $\mathcal{I}$ *satisfies* a concept assertion if $a^{\mathcal{I}} \in [C]^{\mathcal{I}}$. It *satisfies* a role assertion if $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$. We say that an interpretation $\mathcal{I}$ *satisfies* (is a *model* of) an ABox if it satisfies all the assertions in the ABox. An ABox is *consistent* if there is an interpretation that satisfies it.

We can easily reduce concept satisfiability to ABox satisfiability. Take any individual name a. A concept C is satisfiable if and only if the ABox $\mathcal{A} = \{C(a)\}$ is satisfiable.

A *knowledge base* is a pair $\mathcal{K} = (\mathcal{A}, \mathcal{T})$, where $\mathcal{A}$ is an ABox and $\mathcal{T}$ is a TBox.

Definition 95 (translation into first-order logic)**.** We define the *translation* of an $\mathcal{ALC}$-concept to a first-order formula over the signature that contains the unary predicate symbols N_C and the binary predicate symbols N_R.

$$\mathsf{ST}_{xy}[A] = A(x) \qquad \mathsf{ST}_{xy}[C \sqcap D] = \mathsf{ST}_{xy}[C] \wedge \mathsf{ST}_{xy}[D]$$
$$\mathsf{ST}_{xy}[\neg C] = \neg\mathsf{ST}_{xy}[C] \qquad \mathsf{ST}_{xy}[\exists r.C] = \exists y(r(x,y) \wedge \mathsf{ST}_{yx}[C])$$

It is clear that the translation $\mathsf{ST}_{xy}[C]$ of any $\mathcal{ALC}$-concept C has exactly one free variable, namely x. We will also write $\mathsf{ST}_x[C]$ to mean any $\mathsf{ST}_{xy}[C]$ for $y \neq x$.

Similarly, we can translate $\mathcal{ALC}$-concept inclusion axioms, $\mathcal{ALC}$-TBoxes, and $\mathcal{ALC}$-ABoxes into first-order sentences.

$$\begin{aligned}
\mathsf{ST}[C \dot{\sqsubseteq} D] &= \forall x(\mathsf{ST}_x[C] \to \mathsf{ST}_x[D]) \\
\mathsf{ST}[\mathcal{T}] &= \bigwedge_{C \dot{\sqsubseteq} D \in \mathcal{T}} \mathsf{ST}[C \dot{\sqsubseteq} D] \\
\mathsf{ST}[C(a)] &= [\mathsf{ST}_x[C]]\{a/x\} \\
\mathsf{ST}[r(a,b)] &= r(a,b) \\
\mathsf{ST}[\mathcal{A}] &= \bigwedge_{C(a) \in \mathcal{A}} \mathsf{ST}[C(a)] \wedge \bigwedge_{r(a,b) \in \mathcal{A}} \mathsf{ST}[r(a,b)]
\end{aligned}$$

Remark 96. With an easy induction, we show that for every $\mathcal{ALC}$-concept C, and every $d \in \Delta^{\mathcal{I}}$, $d \in [C]^{\mathcal{I}} \iff \mathcal{I}, x \mapsto d \models \mathsf{ST}_x[C]$. It follows that

$$\mathcal{I} \text{ satisfies } C \iff \text{there is } d \in \Delta^{\mathcal{I}} \text{ s.t. } \mathcal{I}, x \mapsto d \models \mathsf{ST}_x[C].$$

It is also very straightforward to prove the following equivalences.

$$\mathcal{I} \text{ satisfies } C \mathrel{\dot\sqsubseteq} D \iff \mathcal{I} \models \mathsf{ST}[C \mathrel{\dot\sqsubseteq} D] \qquad \mathcal{I} \text{ satisfies } C(a) \iff \mathcal{I} \models \mathsf{ST}[C(a)]$$

$$\mathcal{I} \text{ satisfies } \mathcal{T} \iff \mathcal{I} \models \mathsf{ST}[\mathcal{T}] \qquad \mathcal{I} \text{ satisfies } r(a,b) \iff \mathcal{I} \models \mathsf{ST}[r(a,b)]$$

$$\mathcal{I} \text{ satisfies } \mathcal{A} \iff \mathcal{I} \models \mathsf{ST}[\mathcal{A}]$$

$$\mathcal{I} \text{ satisfies } C \text{ w.r.t. } \mathcal{T} \iff \mathcal{I} \models \mathsf{ST}[\mathcal{T}] \text{ and there is } d \in \Delta^{\mathcal{I}} \text{ s.t. } \mathcal{I}, x \mapsto d \models \mathsf{ST}_x[C]$$

$$\mathcal{I} \text{ satisfies } \mathcal{A} \text{ w.r.t. } \mathcal{T} \iff \mathcal{I} \models \mathsf{ST}[\mathcal{T}] \wedge \mathsf{ST}[\mathcal{A}]$$

It follows that the satisfiability and consistency problem of this description logic can be reduced to the satisfiability problem of first-order logic.

10.2. **Temporalizing Description Logics.** There several options available when temporalizing a description logic. One may allow temporal operators to be applied to concepts, roles, TBoxes, ABoxes, or any combination of those. We will only consider linear-time temporal operators, such as $\bigcirc$, $\bullet$, F (sometime in the future), P (sometime in the past), $\mathscr{U}$ (until), $\mathscr{S}$ (since), $\Box$ (at all moments), $\Diamond$ (at some time), etc.

Definition 97 (temporal concepts)**.** Let $\mathcal{CL}$ be a non-temporal concept language that we want to enrich with linear-time temporal operators. The augmented set of concepts is denoted by $\mathcal{CL}_{ops}$, where *ops* is the set of temporal operators we include in the language. We will write, for example, $\mathcal{ALC}_{\mathscr{U}\mathscr{S}}$ to mean the set of concepts that are built using the usual operators of $\mathcal{ALC}$ and the temporal operators $\mathscr{U}$ and $\mathscr{S}$. Semantics for $\mathcal{CL}_{ops}$-concepts is defined over a *temporal interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\cdot^{\mathcal{I}}$ maps each atomic concept A to a subset $A^{\mathcal{I}} \subseteq \mathbb{N} \times \Delta^{\mathcal{I}}$ and each atomic r role to a subset $r^{\mathcal{I}} \subseteq \mathbb{N} \times \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. Let us see, for example, how we would extend the interpretation function to $\mathcal{ALC}_{\mathscr{U}\mathscr{S}}$-concepts.

$$[\exists r.C]^{\mathcal{I}} = \big\{(u,x) \in \mathbb{N} \times \Delta^{\mathcal{I}} \mid \text{there is } y \in \Delta^{\mathcal{I}} \text{ s.t. } (u,x,y) \in [r]^{\mathcal{I}} \text{ and } (u,y) \in [C]^{\mathcal{I}}\big\}$$

$$[C \mathscr{U} D]^{\mathcal{I}} = \big\{(u,x) \in \mathbb{N} \times \Delta^{\mathcal{I}} \mid \text{there is } v \in \mathbb{N} \text{ s.t. } u < v, \\ (v,x) \in [D]^{\mathcal{I}}, \text{ and } (w,x) \in [C]^{\mathcal{I}} \text{ for all } w \in (u,v)\big\}$$

$$[C \mathscr{S} D]^{\mathcal{I}} = \big\{(u,x) \in \mathbb{N} \times \Delta^{\mathcal{I}} \mid \text{there is } v \in \mathbb{N} \text{ s.t. } v < u, \\ (v,x) \in [D]^{\mathcal{I}}, \text{ and } (w,x) \in [C]^{\mathcal{I}} \text{ for all } w \in (v,u)\big\}$$

The atomic cases and the cases of the Boolean operators are defined as for $\mathcal{ALC}$-concepts.

If C, D are $\mathcal{CL}_{ops}$-concepts, then $C \mathrel{\dot\sqsubseteq} D$ is a $\mathcal{CL}_{ops}$-concept *inclusion axiom.* An $\mathcal{CL}_{ops}$-TBox (non-temporal) is a finite set of $\mathcal{CL}_{ops}$-concept inclusion axioms. If C is a $\mathcal{CL}_{ops}$-concept and a is an individual name, then $C(a)$ is a $\mathcal{CL}_{ops}$*-concept assertion.* An $\mathcal{CL}_{ops}$-ABox (non-temporal) is a finite set of $\mathcal{CL}_{ops}$-concept assertions and of role assertions.

Definition 98. Let $\mathcal{CL}$ be a non-temporal concept language, and *ops* a set of linear-time temporal operators.

- An interpretation $\mathcal{I}$ *satisfies* a $\mathcal{CL}_{ops}$-concept C at a moment u if there is a point x in $\Delta^{\mathcal{I}}$ such that $(u,x) \in [C]^{\mathcal{I}}$.
- A $\mathcal{CL}_{ops}$-concept C is *satisfiable* if there is an interpretation $\mathcal{I}$ and a moment u such that $\mathcal{I}$ satisfies C at u.
- An interpretation $\mathcal{I}$ *satisfies* a $\mathcal{CL}_{ops}$-concept inclusion axiom $C \mathrel{\dot\sqsubseteq} D$ if $[C]^{\mathcal{I}} \subseteq [D]^{\mathcal{I}}$.
- An interpretation $\mathcal{I}$ *satisfies* a $\mathcal{CL}_{ops}$-TBox if it satisfies all the inclusion axioms in it.
- A $\mathcal{CL}_{ops}$-TBox $\mathcal{T}$ is *satisfiable* if there is an interpretation that satisfies it.
- An interpretation $\mathcal{I}$ *satisfies a* $\mathcal{CL}_{ops}$*-concept* C *w.r.t. a* $\mathcal{CL}_{ops}$-TBox $\mathcal{T}$ if $\mathcal{I}$ satisfies $\mathcal{T}$ and also satisfies C at some moment.

– A $\mathcal{CL}_{ops}$-concept C is *satisfiable w.r.t. a* $\mathcal{CL}_{ops}$*-*TBox $\mathcal{T}$ if there an interpretation that satisfies C w.r.t. $\mathcal{T}$.
– An interpretation $\mathcal{I}$ *satisfies* a $\mathcal{CL}_{ops}$-concept assertion $C(a)$ at a moment u if $(u, a^{\mathcal{I}}) \in [C]^{\mathcal{I}}$.
– An interpretation $\mathcal{I}$ *satisfies* a role assertion $r(a, b)$ at moment u if $(u, a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$.
– An interpretation $\mathcal{I}$ *satisfies* a $\mathcal{CL}_{ops}$-ABox at a moment u if it satisfies all the assertions in it at u.
– A $\mathcal{CL}_{ops}$-ABox is *consistent* if there is an interpretation that satisfies it at some moment.
– A $\mathcal{CL}_{ops}$-ABox $\mathcal{A}$ is *consistent w.r.t. a* $\mathcal{CL}_{ops}$-TBox $\mathcal{T}$ if there is an interpretation $\mathcal{I}$ that satisfies $\mathcal{T}$ and there is also a moment u such that $\mathcal{I}$ satisfies $\mathcal{A}$ at u.

Definition 99 (translation to first-order temporal logic)**.** Let $\mathcal{CL}$ be a non-temporal concept language and *ops* be a set of temporal operators. We extend the translation of $\mathcal{CL}$-concepts so that $\mathcal{CL}_{ops}$ is covered. For example,

$$\mathsf{ST}_{xy}[C \mathscr{U} D] = \mathsf{ST}_{xy}[C] \mathscr{U} \mathsf{ST}_{xy}[D] \qquad \mathsf{ST}_{xy}[C \mathscr{S} D] = \mathsf{ST}_{xy}[C] \mathscr{S} \mathsf{ST}_{xy}[D] \ .$$

We translate $\mathcal{CL}_{ops}$-TBoxes and $\mathcal{CL}_{ops}$-ABoxes as in Definition 95 with the only difference being

$$\mathsf{ST}[C \mathrel{\dot\sqsubseteq} D] = \Box \forall x (\mathsf{ST}_x[C] \to \mathsf{ST}_x[D]) \ .$$

That is, for $\mathcal{CL}_{ops}$-TBoxes we also have universal quantification over time.

Remark 100. Let C be a $\mathcal{CL}_{ops}$-concept and $\mathcal{I}$ a temporal interpretation. For any moment u and any domain element d, $(u, d) \in [C]^{\mathcal{I}} \iff \mathcal{I}, x \mapsto d, u \models \mathsf{ST}_x[C]$. A temporal interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ can be seen as a first-order temporal structure $\mathcal{I} = (\langle \mathbb{N}, < \rangle, \Delta^{\mathcal{I}}, I)$, where

$$A^{I_u} = \{ d \mid (u, d) \in [A]^{\mathcal{I}} \} \qquad r^{I_u} = \{ (d_1, d_2) \mid (u, d_1, d_2) \in [r]^{\mathcal{I}} \} \qquad a^{I_u} = a^{\mathcal{I}} \ .$$

When we allow the application of temporal operators to concept inclusions, we speak of *temporal* TBox*es.*

Definition 101 (temporal TBoxes)**.** Let $\mathcal{CL}$ be a (temporal or non-temporal) concept language and *ops* a set of temporal operators. We define the set of $\mathcal{CL}$-*ops*-TBoxes as the smallest set that satisfies the following conditions.

- If C, D are $\mathcal{CL}$-concepts, then $C \mathrel{\dot\sqsubseteq} D$ is a $\mathcal{CL}$-*ops*-TBox.
- If ϕ is a $\mathcal{CL}$-*ops*-TBox, then $\neg\phi$ is a $\mathcal{CL}$-*ops*-TBox.
- If ϕ, ψ are $\mathcal{CL}$-*ops*-TBoxes, then $(\phi \wedge \psi)$ is a $\mathcal{CL}$-*ops*-TBox.
- For any unary operator $\circ \in$ *ops*, if ϕ is a $\mathcal{CL}$-*ops*-TBox, then $\circ\phi$ is a $\mathcal{CL}$-*ops*-TBox.
- For any binary operator $\otimes \in$ *ops*, if ϕ, ψ are $\mathcal{CL}$-*ops*-TBoxes, then $(\phi \otimes \psi)$ is a $\mathcal{CL}$-*ops*-TBox.

The above definition allows us to choose independently the temporal operators that will be applied on concepts and the temporal operators that are applied on TBoxes. For example, we can have a $\mathcal{ALC}_{\mathsf{FP}}$-$\mathscr{U}\mathscr{S}$-$\mathsf{TBox}$.

We define *truth* of a $\mathcal{CL}$-*ops*-TBox under an interpretation $\mathcal{I}$ at a moment u inductively as follows.

$$\begin{aligned} \mathcal{I}, u \models C \mathrel{\dot\sqsubseteq} D &\iff \text{for all } x \in \Delta^{\mathcal{I}},\ (u, x) \in [C]^{\mathcal{I}} \implies (u, x) \in [D]^{\mathcal{I}} \\ \mathcal{I}, u \models \neg\phi &\iff \mathcal{I}, u \not\models \phi \\ \mathcal{I}, u \models (\phi \wedge \psi) &\iff \mathcal{I}, u \models \phi \text{ and } \mathcal{I}, u \models \psi \end{aligned}$$

For $\mathcal{CL}$-*ops*-TBoxes that start with a temporal operator, truth is defined in the obvious way. If, for example, *ops* were equal to $\{\mathscr{U}, \mathscr{S}\}$, we would have

$$\begin{aligned} \mathcal{I}, u \models (\phi \mathscr{U} \psi) &\iff \text{there is } v \in \mathbb{N} \text{ s.t. } u < v,\ \mathcal{I}, v \models \psi, \text{ and } \mathcal{I}, w \models \phi \text{ for all } w \in (u, v) \\ \mathcal{I}, u \models (\phi \mathscr{S} \psi) &\iff \text{there is } v \in \mathbb{N} \text{ s.t. } v < u,\ \mathcal{I}, v \models \psi, \text{ and } \mathcal{I}, w \models \phi \text{ for all } w \in (v, u) \ . \end{aligned}$$

Definition 102. Let $\mathcal{CL}$ be a temporal or non-temporal concept language and *ops* a set of temporal operators.

– We say that the intepretation $\mathcal{I}$ *satisfies* the $\mathcal{CL}$-*ops*-TBox ϕ at moment u, if $\mathcal{I}, u \models \phi$.

- A $\mathcal{CL}$-*ops*-TBox ϕ is said to be *satisfiable* if there is an interpretation $\mathcal{I}$ and a moment u such that $\mathcal{I}$ satisfies ϕ at u.
- An interpretation $\mathcal{I}$ *satisfies* a $\mathcal{CL}$-concept C w.r.t. a $\mathcal{CL}$-*ops*-TBox ϕ at a moment u if $\mathcal{I}$ satisfies both C and ϕ at u.
- A $\mathcal{CL}$-concept C is said to be *satisfiable* w.r.t. a $\mathcal{CL}$-*ops*-TBox ϕ if there is an interpretation $\mathcal{I}$ and a moment u such that $\mathcal{I}$ satisfies C w.r.t. ϕ at u.

Observe that non-temporal TBoxes are interpreted *globally*, in the sense that the inclusions have to hold at every moment. Temporal TBoxes, on the other hand, are interpreted *locally* at every moment of time.

It is easy to see that a non-temporal $\mathcal{CL}$-TBox $\mathcal{T}$ is satisfied by an interpretation $\mathcal{I}$ if and only if the (temporal) $\mathcal{CL}$-$\Box$-TBox

$$\phi = \Box \bigwedge_{C \dot\sqsubseteq D \in \mathcal{T}} C \dot\sqsubseteq D$$

is satisfied by $\mathcal{I}$ at some (at all) moment(s).

A $\mathcal{CL}$-concept C is satisfiable w.r.t. a $\mathcal{CL}$-*ops*-TBox ϕ if and only if $\neg(C \dot\sqsubseteq \bot) \wedge \phi$ is satisfiable.

Definition 103 (temporal ABoxes)**.** Let $\mathcal{CL}$ be a (temporal or non-temporal) concept language and *ops* a set of temporal operators. We define the set of $\mathcal{CL}$-*ops*-ABoxes as the smallest set that satisfies the following conditions.

- If C is a $\mathcal{CL}$-concept and a is an individual name, then $C(a)$ is a $\mathcal{CL}$-*ops*-ABox.
- If r is a role and a, b are individual names, then $r(a, b)$ is a $\mathcal{CL}$-*ops*-ABox.
- If ϕ is a $\mathcal{CL}$-*ops*-ABox, then $\neg\phi$ is a $\mathcal{CL}$-*ops*-ABox.
- If ϕ, ψ are $\mathcal{CL}$-*ops*-ABoxes, then $(\phi \wedge \psi)$ is a $\mathcal{CL}$-*ops*-ABox.
- For any unary operator $\circ \in ops$, if ϕ is a $\mathcal{CL}$-*ops*-ABox, then $\circ\phi$ is a $\mathcal{CL}$-*ops*-ABox.
- For any binary operator $\otimes \in ops$, if ϕ, ψ are $\mathcal{CL}$-*ops*-ABoxes, then $(\phi \otimes \psi)$ is a $\mathcal{CL}$-*ops*-ABox.

Again, we observe that the definition allows us to choose independently the temporal operators for the concepts and the temporal operators for the ABoxes. For example, we can have a $\mathcal{ALC}_{\mathscr{U}\mathscr{S}}$-F$\bigcirc$P$\bullet$-ABox.

Truth of a $\mathcal{CL}$-*ops*-ABox under an interpretation $\mathcal{I}$ at a moment u is defined as for temporal TBoxes. The base cases are defined as one would expect:

$$\mathcal{I}, u \models C(a) \iff (u, a^{\mathcal{I}}) \in [C]^{\mathcal{I}} \qquad\qquad \mathcal{I}, u \models r(a, b) \iff (u, a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}\ .$$

Definition 104. Let $\mathcal{CL}$ be a (temporal or non-temporal) concept language and ops, ops_1, ops_2 be sets of temporal operators.

- An interpretation $\mathcal{I}$ *satisfies* the $\mathcal{CL}$-*ops*-ABox ϕ at moment u if $\mathcal{I}, u \models \phi$.
- A $\mathcal{CL}$-*ops*-ABox ϕ is *consistent* if there is an interpretation $\mathcal{I}$ and a moment u such that $\mathcal{I}$ satisfies ϕ at u.
- A $\mathcal{CL}$-*ops*-ABox ϕ is *consistent w.r.t. a* $\mathcal{CL}$-TBox $\mathcal{T}$ if there is an interpretation $\mathcal{I}$ that satisfies $\mathcal{T}$ and there is also a moment u such that $\mathcal{I}$ satisfies ϕ at u.
- A $\mathcal{CL}$-ops_1-ABox ϕ is *consistent w.r.t. a* $\mathcal{CL}$-ops_2-TBox ψ if there is an interpretation $\mathcal{I}$ and a moment u such that $\mathcal{I}$ satisfies both ϕ and ψ at u.

Definition 105 (temporal knowledge bases)**.** We generalize temporal TBoxes and temporal ABoxes, so that axioms, concept assertions, and role assertions are treated as atoms that can be combined with Boolean and temporal operators.

Let $\mathcal{CL}$ be a (temporal or non-temporal) concept language and *ops* a set of temporal operators. We define the set of $\mathcal{CL}$-*ops*-formulas (we will use the terms formula and knowledge base interchangeably) to be the smallest set that satisfies the following.

- If C, D are $\mathcal{CL}$-concepts, then $C \dot\sqsubseteq D$ is an atomic $\mathcal{CL}$-*ops*-formula.
- If C is a $\mathcal{CL}$-concept and a is an individual name, then $C(a)$ is a $\mathcal{CL}$-*ops*-formula.
- If r is a role and a, b are individual names, then $r(a, b)$ is a $\mathcal{CL}$-*ops*-formula.
- If ϕ is a $\mathcal{CL}$-*ops*-formula, then $\neg\phi$ is a $\mathcal{CL}$-*ops*-formula.

- If ϕ, ψ are $\mathcal{CL}$-*ops*-formulas, then $(\phi \wedge \psi)$ is a $\mathcal{CL}$-*ops*-formula.
- For any unary operator $\circ \in ops$, if ϕ is a $\mathcal{CL}$-*ops*-formula, then $\circ\phi$ is a $\mathcal{CL}$-*ops*-formula.
- For any binary operator $\otimes \in ops$, if ϕ, ψ are $\mathcal{CL}$-*ops*-formulas, then $(\phi \otimes \psi)$ is a $\mathcal{CL}$-*ops*-formula.

Truth is defined as for temporal TBoxes and temporal ABoxes. An interpretation $\mathcal{I}$ *satisfies* a temporal knowledge base ϕ at a moment u if $\mathcal{I}, u \models \phi$. A temporal knowledge base is said to be *consistent* if there is an interpretation that satisfies it at some moment.

Theorem 106. Let $\mathcal{CL}$ be the concept language that involves only the constructors $\neg$ and $\sqcap$. The satisfiability problem for $\mathcal{CL}_{\mathsf{G}}$-G-TBoxes is EXPSPACE-hard. The same holds for the finite satisfiability problem. [30]

Proof. $\mathcal{CL}_{\mathsf{G}}$-G-TBoxes are expressive enough to encode the 2^n-corridor tiling problem, which is known to be EXPSPACE-complete for n encoded in unary. $\square$

10.3. **Temporal Description Logics with Fixpoint Operators.** We can use the complexity results obtained in Section 7 to get immediately upper complexity bounds for various temporal description logics with fixpoint operators. First, we will show how we augment $\mathcal{ALC}$ with fixpoint operators.

Definition 107 ($\mathcal{ALC}_\mu$-concepts,$\mathcal{ALC}_\mu$-μ-formulas)**.** Fix a set $N_C = \{A_1, A_2, \ldots\}$ of atomic concepts, a set $N_R = \{r_1, r_2, \ldots\}$ of atomic roles, a set $\mathcal{X} = \{X_1, X_2, \ldots\}$ of concept fixpoint variables, a set $\mathcal{Y} = \{Y_1, Y_2, \ldots\}$ of formula fixpoint variables, and a set $N_I = \{a_1, a_2, \ldots\}$ of individual names. The following grammar generates the set of $\mathcal{ALC}_\mu$*-concepts.*

$$C, D ::= A \mid X \mid \neg C \mid (C \sqcap D) \mid \exists r.C \mid \bigcirc C \mid \CIRCLE C \mid \mu X.C$$

For a concept $\mu X.C$, we have the restriction that the free occurrences of the concept fixpoint variable X in C are only positive. Given a temporal interpretation $\mathcal{I}$, which maps each concept fixpoint variable to a subset of $\mathbb{N} \times \Delta^{\mathcal{I}}$, we define

$$\begin{aligned}
[\bigcirc C]^{\mathcal{I}} &= \big\{(v, d) \in \mathbb{N} \times \Delta^{\mathcal{I}} \mid (v+1, d) \in [C]^{\mathcal{I}}\big\} \\
[\CIRCLE C]^{\mathcal{I}} &= \big\{(v, d) \in \mathbb{N} \times \Delta^{\mathcal{I}} \mid v > 0 \text{ and } (v-1, d) \in [C]^{\mathcal{I}}\big\} \\
[\mu X.C]^{\mathcal{I}} &= \text{the least fixpoint of the function } f(S) = [C]^{\mathcal{I}[X \mapsto S]} \ .
\end{aligned}$$

The set of $\mathcal{ALC}_\mu$-μ*-formulas* is defined by the grammar

$$\phi, \psi ::= C \mathrel{\dot\sqsubseteq} D \mid Y \mid \neg\phi \mid (\phi \wedge \psi) \mid \bigcirc\phi \mid \CIRCLE\psi \mid \mu Y.\phi \ ,$$

where C, D are $\mathcal{ALC}_\mu$-concepts. As before, we require that for a formula $\mu Y.\phi$, the free occurrences of the formula fixpoint variable Y in ϕ are positive. Given a temporal interpretation $\mathcal{I}$ and a fixpoint variable assignment $g : \mathcal{Y} \to \wp(\mathbb{N})$, we define

$$\begin{aligned}
\mathcal{I}, g, u \models Y &\iff u \in g(Y) \\
\mathcal{I}, g, u \models \bigcirc\phi &\iff \mathcal{I}, g, u+1 \models \phi \\
\mathcal{I}, g, u \models \CIRCLE\phi &\iff u > 0 \text{ and } \mathcal{I}, g, u-1 \models \phi \\
\mathcal{I}, g, u \models \mu Y.\phi &\iff u \in \mathit{lfp}(f), \text{ where } f(S) = \{v \in \mathbb{N} \mid \mathcal{I}, g[Y \mapsto S], v \models \phi\}
\end{aligned}$$

An $\mathcal{ALC}_\mu$-μ*-sentence* is a $\mathcal{ALC}_\mu$-μ-formula with no free occurrences of fixpoint variables.

Definition 108 (monodic concepts)**.** We define the set $\mathcal{ALC}_{1\mu}$ of monodic $\mathcal{ALC}_\mu$-concepts as the subset of $\mathcal{ALC}_{1\mu}$ in which we do not allow any concept fixpoint variables to occur free in the scope of $\exists$. Observe that in $\mathcal{ALC}_{1\mu}$-μ-formulas, there is not the risk of having a formula fixpoint variable occur in the scope of quantification, since we only have implicit universal quantification in inclusion axioms.

Theorem 109.

- The satisfiabiliy problem for $\mathcal{ALC}_{1\mu}$-μ-sentences is EXPSPACE-complete.
- The finite satisfiability problem for $\mathcal{ALC}_{1\mu}$-μ-sentences is EXPSPACE-complete.

Proof. Consider the translation of a $\mathcal{ALC}_{1\mu}$-μ-formula in first-order temporal logic. The first-order part of the translation is clearly in the two-variable fragment of first-order logic, the satisfiability problem of which is in NEXPTIME [23], which is contained in EXPSPACE. The first claim follows from Theorem 63 and Theorem 106. For the second claim, we note that the two-variable fragment has the finite model property [45]. $\square$

11. Conclusion

In the case of full first-order linear-time temporal logic, which is the combination of a RE-complete logic with a PSPACE-complete logic, practical applications are hopeless due to its high undecidability (more specifically Σ_1^1-completeness). The monodic restriction manages to bring down several monodic fragments to the class of recursive languages. Some monodic fragments over the naturals are even in exponential space. The question naturally arises as to how far this idea can be pushed to get more expressivity.

A major part of our work here is devoted to showing that we can extend the temporal part with fixpoint operators and still keep appropriately restricted monodic fragments over the naturals within exponential space (Theorem 63). The same result is shown for the problem of finite satisfiability (Theorem 66). We also obtain upper complexity bounds — 2EXPTIME and EXPSPACE for the bounded-variable or bounded-arity case — for the monodic packed fragment with temporal fixpoint operators over the naturals (Theorem 77). 2EXPTIME-completeness (and EXPSPACE-completeness respectively) follows immediately. We then proceed to add domain-side least fixpoint operators to monodic guarded fragments over the naturals that already have temporal fixpoint operators and show similar complexity results (Theorem 90).

An interesting question is whether decidability is preserved if we relax the monodic restriction. By requiring that any subformula beginning with a quantifier has no free fixpoint variables, we do not allow any "moving from one run to another" while evaluating fixpoints. Can we loosen this requirement and still stay in the realm of the decidable?

In a different direction, one might want to consider branching time logics. There exist decidability results for monodic CTL* [33, 5], but the complexity of these decidable fragments is not known yet.

The case of dense time is also interesting. For rational time, we know that the monodic fragments of first-order temporal logic with fixpoint operators and decidable first-order parts are decidable [13], but there are no complexity results. For real time, there are no decidability results for such monodic fragments, even without considering temporal fixpoint operators.

An important question left open in Section 9 is whether we can also add greatest domain-side fixpoints to monodic guarded fragments. It is also worth investigating whether the monodicity restriction for domain-side fixpoints can be weakened.

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