Canonical Progress Measures for Parity Games

Konstantinos Mamouras

Department of Computer Science, Cornell University, Ithaca, NY, USA

Email: mamouras@cs.cornell.edu

Abstract—A *progress measure* for a parity game is a labeling $s(\cdot)$ of the vertices of the game that witnesses winning strategies for the players: Player 0 (Even) and Player 1 (Odd). We give a natural definition of a *canonical progress measure* for a parity game G directly, without translating into the μ-calculus. The label $s(u)$ of a vertex u is defined to be the value of an infinitely-played game played on the graph of G.

In order to show the existence of these values, we introduce a *finitely-played version* of the game of duration at most $n(n+1)/2$ moves, where n is the number of vertices. We show that the values for the finitely-played version are the same as the values for the infinitely-played version. This result also implies the existence of optimal strategies for both players (in the infinitely-played version) that use memory of size $O(n^{n(n+1)/2})$.

Without loss of generality we restrict attention to parity games in which Player 0 has a winning strategy from every vertex. We show that Player 0 has a *memoryless* strategy that ensures the canonical progress measure everywhere. For Player 1, optimal strategies are more complex. We consider the special case of 1-solitaire games, i.e. games where only Player 1 has non-trivial moves. For 1-solitaire games, we show that Player 1 must have memory of size $\Omega(n)$ in order to ensure the canonical progress measure. This lower bound extends, of course, to the general case. Moreover, for 1-solitaire games we show that Player 1 has optimal strategies that use memory of size $O(n)$. For the general case, we do not have a matching upper bound for the size of the memory. We improve upon the upper bound previously stated: Player 1 has optimal strategies with memory of size $O(n^n)$.

Our results imply that the canonical progress measure for a parity game G records optimal strategies for Player 0. The same does not seem to be the case, however, for Player 1. We consider this an indication that verifying canonical progress measures for parity games is not easier than finding them.

I. Introduction

A parity game involves two players, Player 0 (or Even) and Player 1 (or Odd). It is played on a directed graph whose vertices are labeled with natural numbers called priorities. The vertices are partitioned into those that belong to Player 0 (0-vertices) and those that belong to Player 1 (1-vertices). A play starts at some vertex where a token is placed. At every step, the player who owns the vertex with the token moves the token to a successor vertex. Thus, an infinite sequence of vertices is formed. Player 0 wins the play if the maximum priority that appears infinitely often is even, otherwise Player 1 wins.

Parity games are memorylessly determined, i.e. for every vertex u some player σ has a memoryless strategy f_σ s.t. every play that starts from u with Player σ playing according to f_σ is won by Player σ. The winning region of Player 0 is the set of vertices from which Player 0 has a winning strategy. Solving parity games amounts to finding the winning region of Player 0. By determinacy, the rest of the vertices are the winning

region of Player 1. The decision version of the problem is: Given a parity game G and a vertex u, does Player 0 have a winning strategy from u? Memoryless determinacy implies that the problem lies in $\mathsf{NP} \cap \mathsf{coNP}$ [1]. Jurdziński has shown that the problem is even contained in $\mathsf{UP} \cap \mathsf{coUP}$ [2].

The importance of finding fast algorithms for solving parity games lies in its polynomial-time equivalence to the problem of model checking the μ-calculus [3], [4]. Despite efforts of the community no polynomial-time algorithm is known for solving parity games. One line of research for designing algorithms for parity games involves the notion of progress measure [5]. Progress measures, introduced by Klarlund and Kozen in [6] where they are called Rabin measures, are annotations of graphs that record progress towards the satisfaction of Rabin conditions. Streett and Emerson used a similar notion, which they called signature [7], to study the μ-calculus.

A progress measure for a parity game is a labelling of the vertices of the game that witnesses winning strategies for the players and hence also the winning regions. The progress measure records progress towards the satisfaction of the parity condition. Walukiewicz considers canonical signature assignments for parity games [8], which are defined by translating the existence of a winning strategy into the μ-calculus and then using the notion of signature by Emerson and Streett [7]. Our definition of the canonical progress measure for a parity game is similar, but does not involve the μ-calculus. The canonical progress measure is unique and records winning strategies for the players that are "good" in the sense of minimizing the progress measure. The progress measure is defined as a labelling $s(\cdot)$ of the vertices so that for a vertex u, $s(u)$ is the value of a game of infinite duration. This is well-defined, because $s(u)$ is shown to be both the least outcome that Player 0 can ensure and the greatest outcome that Player 1 can ensure. This result is also relevant to addressing a question raised by Jurdziński in [2]: Are canonical progress measures unique succinct certificates for parity games? We do not resolve this question here, but we believe that our results provide an indication for a negative answer.

We feel that studying canonical progress measures is important in furthering our understanding of parity games. This is because finding the winning regions, winning strategies, as well as finding some progress measure for a parity game are all equally hard problems. Finding the canonical progress measure is at least as hard. It amounts to finding "good" winning strategies in a precise sense.

II. Summary of results

We define the outcome of an infinite play won by Player 0 (Player 1) to be a function that maps each odd (even) priority to a natural number. We call such a function a *0-signature* (*1-signature*). Consider the lexicographic ordering of these functions, where larger priorities are more significant. Player 0 wants to minimize the outcome and Player 1 to maximize it. We show that for a vertex u both players have strategies that ensure the same outcome $s(u)$ for plays starting from u. In order to show this, we consider a finitely-played version of the game of duration $\leq n(n+1)/2$, where n is the number of vertices. In the finitely-played version, a play ends as soon as a cycle is formed after the first occurrence of the maximum priority that has appeared so far. We say that $s(u)$ is the *value* of u. The values for the finitely-played version are the same as the values for the infinitely-played version. We show this fact using a technique similar to the one used by Ehrenfeucht and Mycielski in [9], where a similar result is established for mean-payoff games. A corollary is that in the infinitely-played version the players have optimal strategies that use memory of size $O(n^{n(n+1)/2})$. The *canonical progress measure* is defined to be the value assignment $s(\cdot)$.

Without loss of generality we study parity games in which Player 0 has a winning strategy from every vertex. First, we establish that Player 0 has a *memoryless* strategy f_0 such that for every vertex u, f_0 ensures outcome $\leq s(u)$ from u. We show an even stronger result: The canonical progress measure records *all* memoryless strategies that ensure the measure from every vertex.

We also study optimal strategies for Player 1. For the simpler case of 1-solitaire games, i.e. games where only Player 1 has non-trivial moves, we show that optimal strategies for Player 1 need memory of size at least $\Omega(n)$. Moreover, optimal strategies can be constructed that use memory of size at most $O(n)$. In order to construct optimal strategies we introduce the notions of *extended outcome* and *extended value*. These notions formalize the idea that Player 1 tries to maximize the outcome in as few steps as possible.

For general parity games, the linear lower bound for the size of the memory also applies. We do not have a matching upper bound. We improve, however, upon the $O(n^{n(n+1)/2})$ upper bound we stated previously. We show that Player 1 has optimal strategies that use memory of size at most $O(n^n)$. Again, constructing these optimal strategies involves the notions of extended outcome and extended value.

III. Conclusion

Let G be a parity game and W_0, W_1 be the winning regions of Player 0 and 1 respectively. Our results for optimal strategies (in general games) are summarized in the following diagram:

W_0	W_1
optimal 0-strategy: memoryless	optimal 1-strategy: memoryless
optimal 1-strategy: $\Omega(n) \leq \|M\| \leq O(n^n)$	optimal 0-strategy: $\Omega(n) \leq \|M\| \leq O(n^n)$

We conjecture that there exist optimal 1-strategies for $G[W_0]$ (and hence also optimal 0-strategies for $G[W_1]$) with memory of size at most $O(n)$, where $G[W]$ denotes the restriction of G to W.

The decision version of the problem of finding the canonical progress measure of a parity game is the following: Given a game G, a vertex u, and a 0-signature t, is it the case that $s(u) \leq t$? Call this problem CANONICAL. We have preliminary results showing that if the above conjecture is true, then CANONICAL lies in NP ∩ coNP.

Let G be a game in which Player 0 wins from every vertex. We have shown that the canonical progress measure records all possible memoryless 0-strategies that ensure the measure. It does not seem, however, that we can read optimal 1-strategies off from the measure. We take this as an indication that verifying canonical progress measures is not easier than finding them, since we might still need to guess an optimal 1-strategy.

References

[1] E. A. Emerson, C. S. Jutla, and A. P. Sistla, "On model-checking for fragments of μ-calculus," in *CAV'93, volume 697 of LNCS*.   Springer-Verlag, 1993, pp. 385–396.

[2] M. Jurdziński, "Deciding the winner in parity games is in UP ∩ co-UP," *Information Processing Letters*, vol. 68, pp. 119–124, 1998.

[3] D. Kozen, "Results on the propositional μ-calculus," in *Proceedings of the 9th Colloquium on Automata, Languages and Programming*, ser. Lecture Notes In Computer Science, vol. 140, 1982, pp. 348–359.

[4] ——, "Results on the propositional μ-calculus," *Theoretical Computer Science*, vol. 27, no. 3, pp. 333–354, 1983.

[5] M. Jurdzinski, "Small progress measures for solving parity games," in *Proceedings of the 17th Annual Symposium on Theoretical Aspects of Computer Science*, ser. STACS '00.   Springer-Verlag, 2000, pp. 290–301.

[6] N. Klarlund and D. Kozen, "Rabin measures and their applications to fairness and automata theory," in *Proceedings of Sixth Annual IEEE Symposium on Logic in Computer Science (LICS 1991)*, 1991, pp. 256–265.

[7] R. S. Streett and E. A. Emerson, "An automata theoretic decision procedure for the propositional mu-calculus," *Information and Computation*, vol. 81, no. 3, pp. 249–264, 1989.

[8] I. Walukiewicz, "Pushdown processes: Games and model-checking," *Information and Computation*, vol. 164, no. 2, pp. 234–263, 2001.

[9] A. Ehrenfeucht and J. Mycielski, "Positional strategies for mean payoff games," *International Journal of Game Theory*, vol. 8, pp. 109–113, 1979.