Week 2: Wednesday, Feb 1

Consider the cubic equation

$$f(x) = x^3 - 2x + c = 0.$$

Describe a general purpose strategy for finding *all* the real roots of this equation for a given c.

Answer: I discussed this in lecture, but I wanted to give a more complete description of it, since I think the ideas are important.

The simplest way I know to solve this problem is to organize the computation around the local minimum and maximum points for f. Setting $f'(x) = 3x^2 - 2$ to zero, we have that there are optima at $z_\pm = \pm\sqrt{2/3}$. The function f is strictly increasing on $(-\infty, z_-]$, strictly decreasing on $[z_-, z_+]$, and strictly increasing again on $[z_+, \infty)$. Thus, we have the following cases:

- $f(z_-) > f(z_+) > 0$: There is one real root on $(-\infty, z_-)$.
- $f(z-) \geq 0 \geq f(z_+)$: There are three real roots, one on $(-\infty, z_-]$, one on $[z_-, z_+]$, and one on $[z_+, \infty)$. In the cases $f(z_-) = 0$ or $f(z_+) = 0$, there is actually a double root at z_- or z_+.
- $0 > f(z_-) > f(z_+)0$: There is one real root on (z_+, ∞).

We can find the root between z_- and z_+ (if there is one) by bisection. What about getting a finite bound for the other two intervals? If we expand f in a Taylor series about z_-, we have

$$f(x) = f(z_-) + 3z_-(x - z_-)^2 + (x - z_-)^3.$$

For $x < z_-$, we have $(x - z_-)^3 < 0$, and so

$$f(x) < f(z_-) + 3z_-(x - z_-)^2.$$

In particular, this means that if f has a zero below z_-, it will be bounded from below by the smaller root of $f(z_-) + 3z_-(x - z_-)^2$, and so will lie in the interval

$$\left[z_- - \sqrt{\frac{-f(z_-)}{3z_-}}, z_- \right].$$

Near a double root, the left end point of this interval becomes an excellent estimate that can be used to start a Newton iteration.

Similarly, a quadratic Taylor series about z_+ provides an upper bound on the root that is greater than z_+, assuming such a root exists.